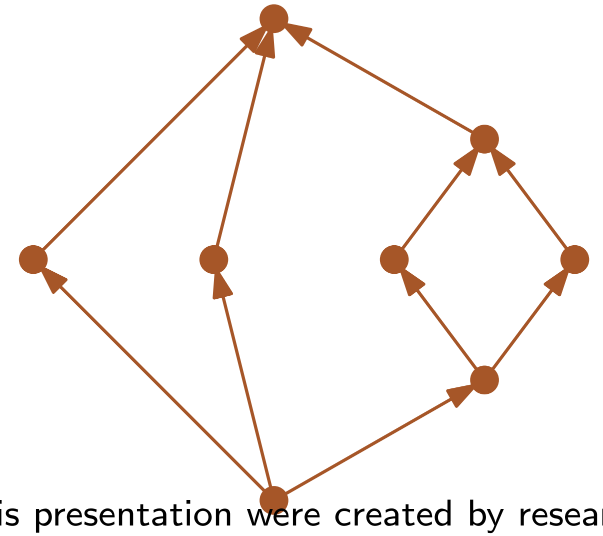
Visualisation of graphs

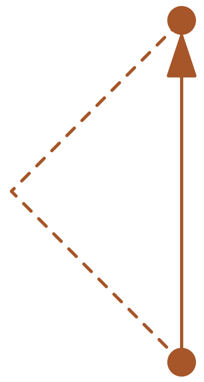
Drawing series-parallel graphs

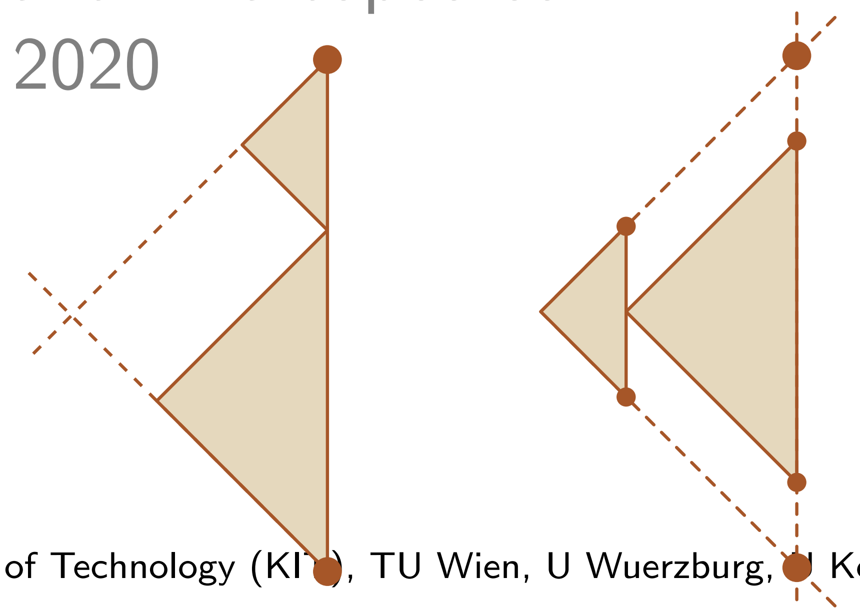
Divide and conquer methods

Antonios Symvonis · Chrysanthi Raftopoulou

Fall semester 2020







The original slides of this presentation were created by researchers at Karlsruhe Institute of Technology (KIT), TU Wien, U Wuerzburg, U Konstanz, ...
The original presentation was modified/updated by A. Symvonis and C. Raftopoulou

Series-parallel graphs

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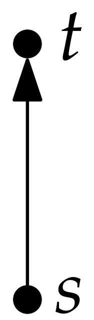
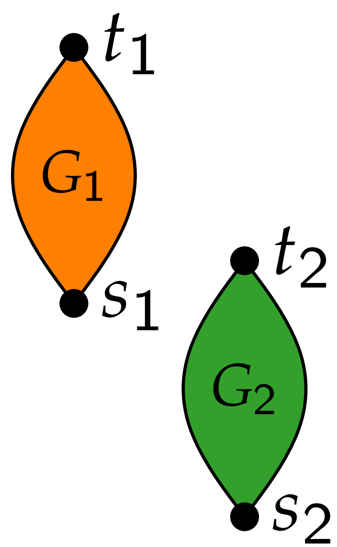
- it consists of two series-parallel graphs G_1, G_2 with sources s_1, s_2 and sinks t_1, t_2 that are combined using one of the following rules:





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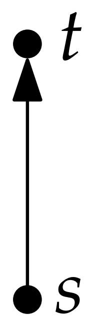
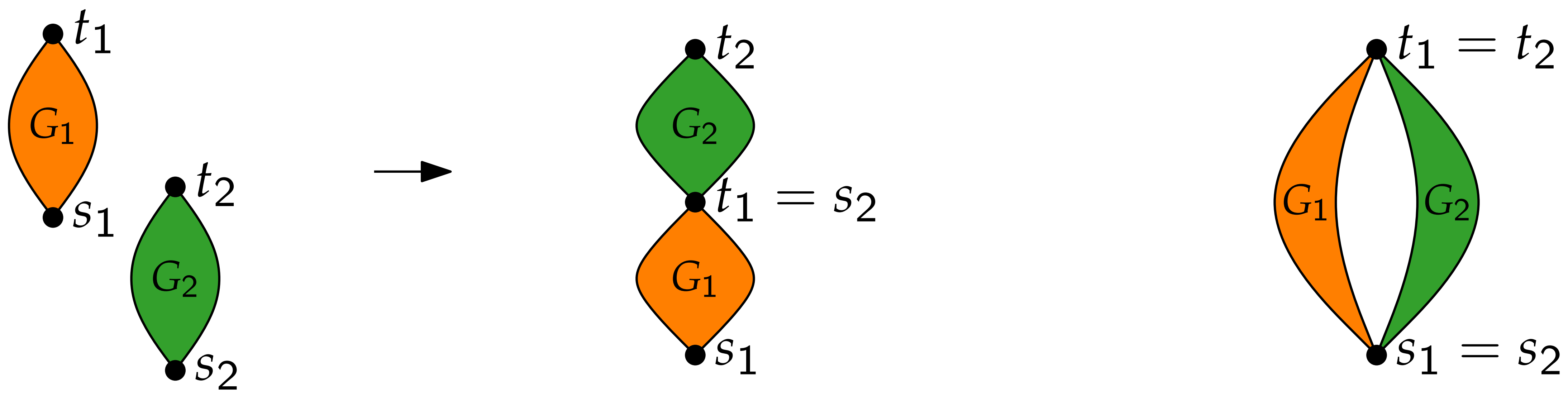


Series composition **Parallel composition**



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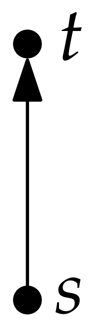

Observations:

- $|E| \leq 2|V| - 3$
- Series-parallel graphs are planar

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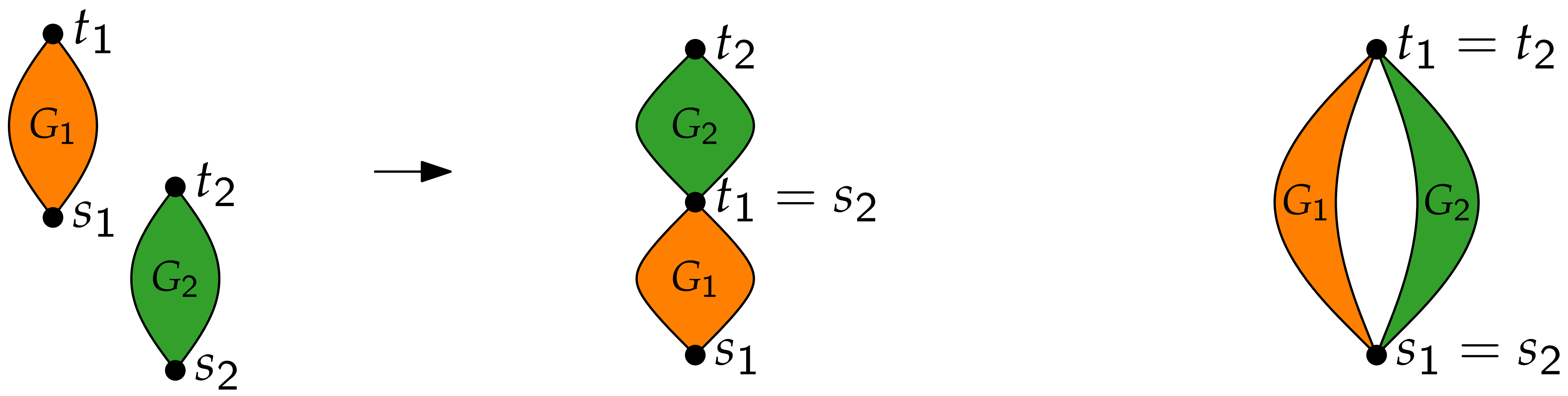

Series-parallel graphs – decomposition tree

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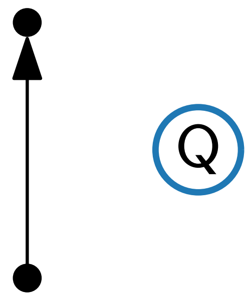

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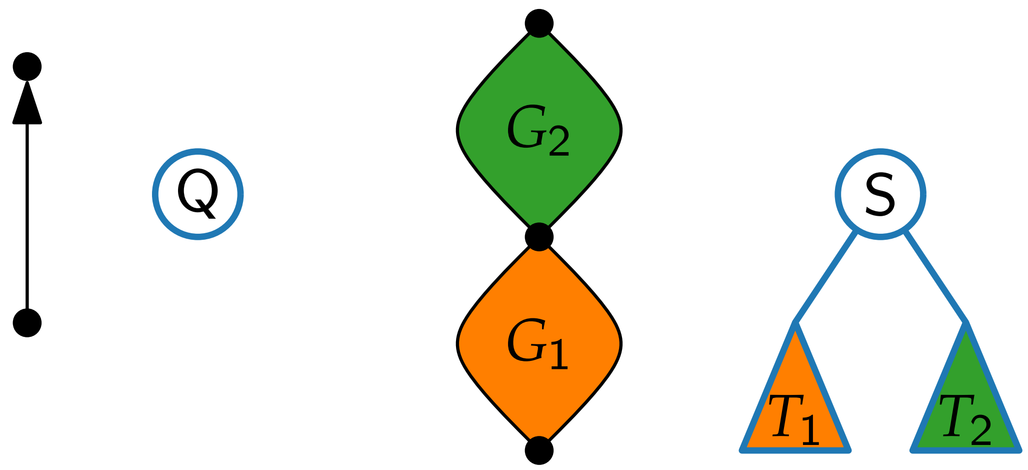

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- A Q-node represents a single edge
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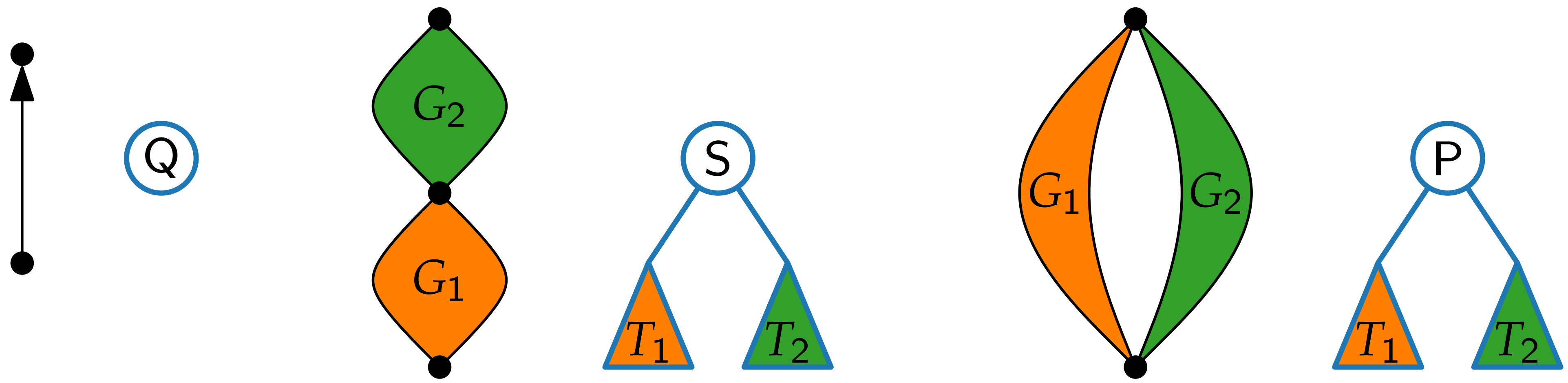
- A P-node represents a parallel composition; its children T_1 and T_2 represent G_1 and G_2



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We further require:

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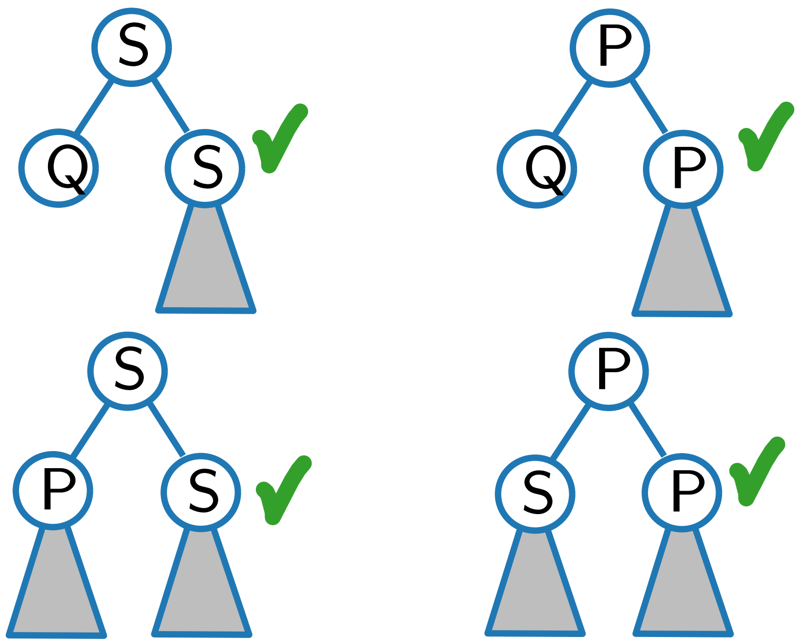

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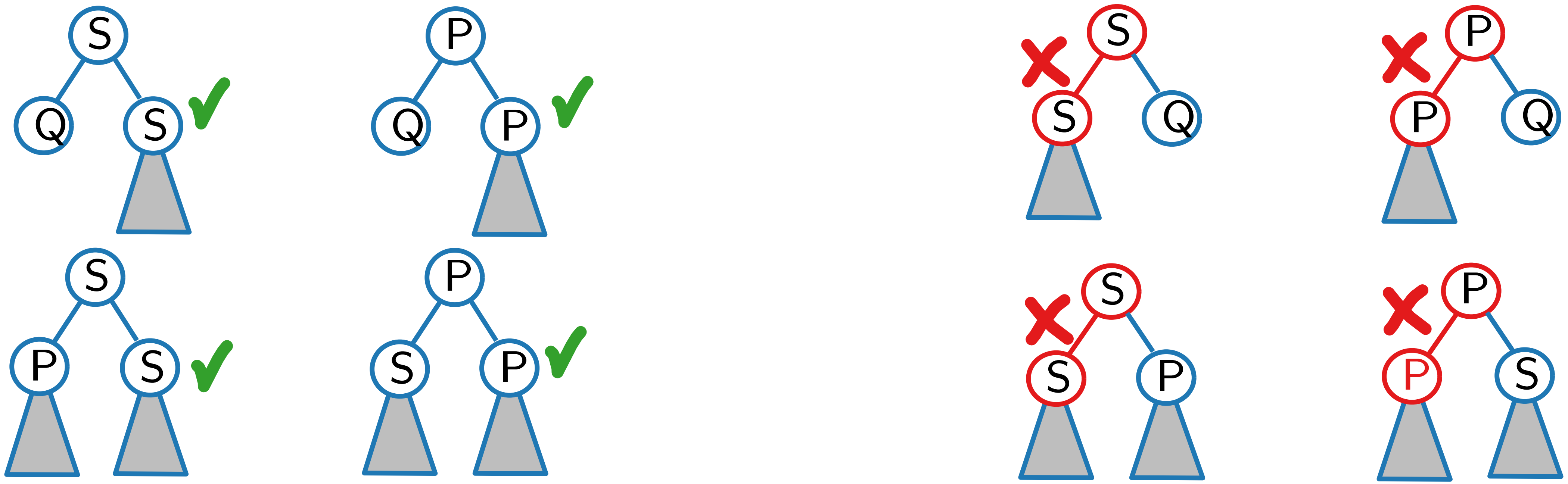

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We further require:

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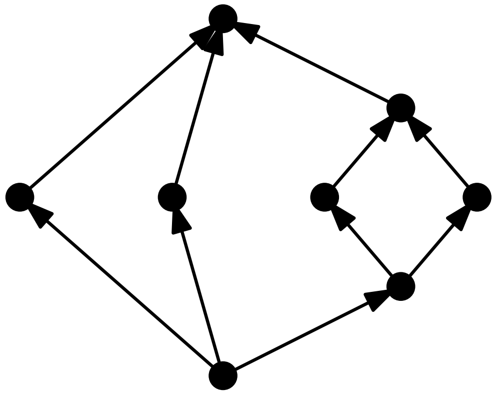
- Unique decomposition tree
- The order of the children (Q or S) define the graph embedding

Series-parallel graphs – decomposition example



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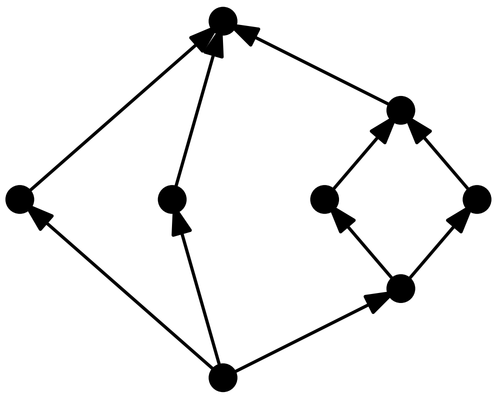

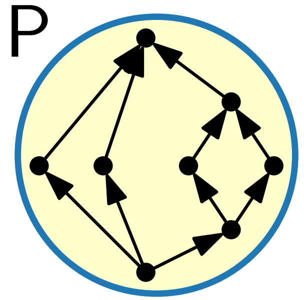

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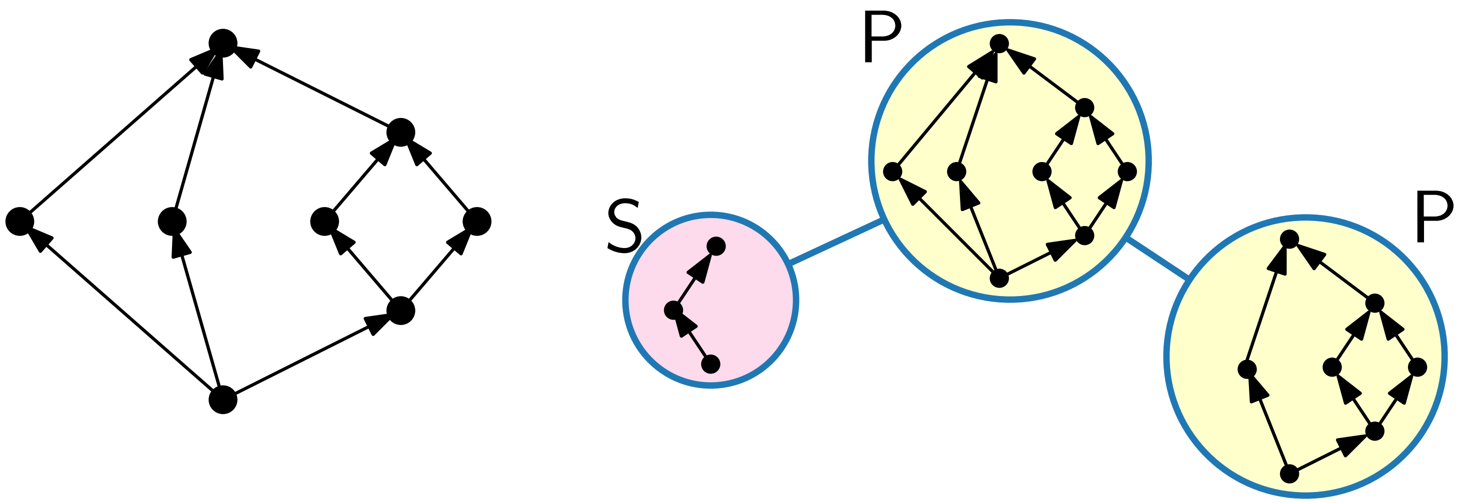

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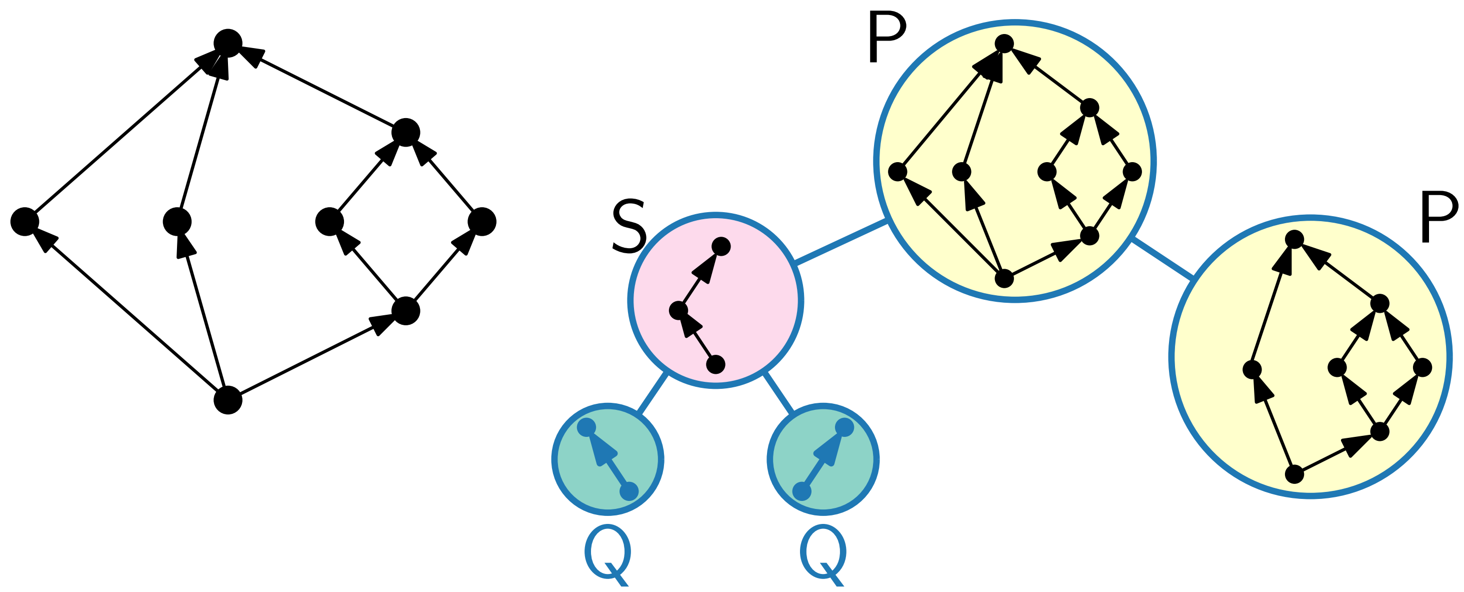

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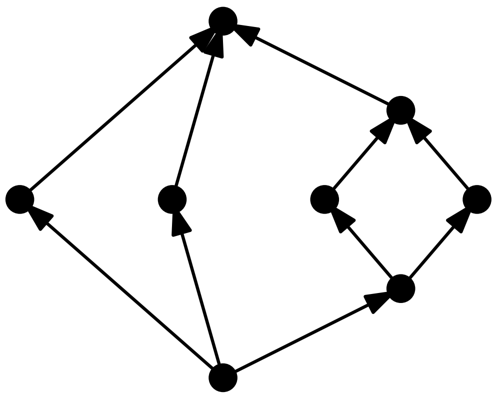

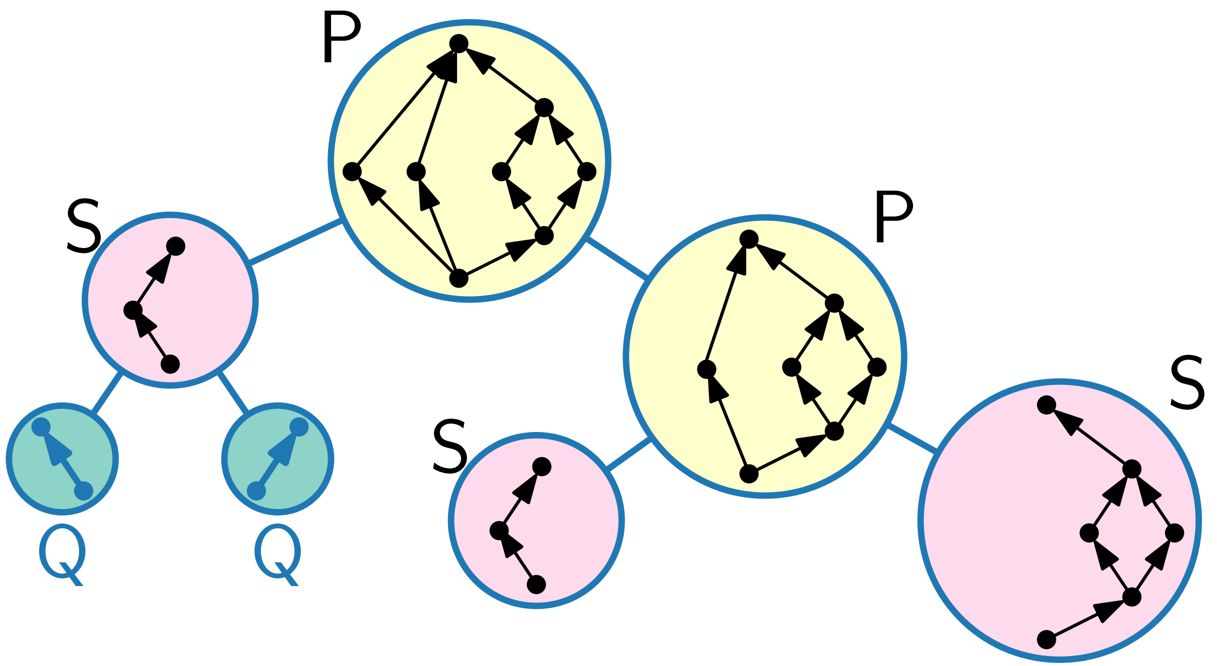

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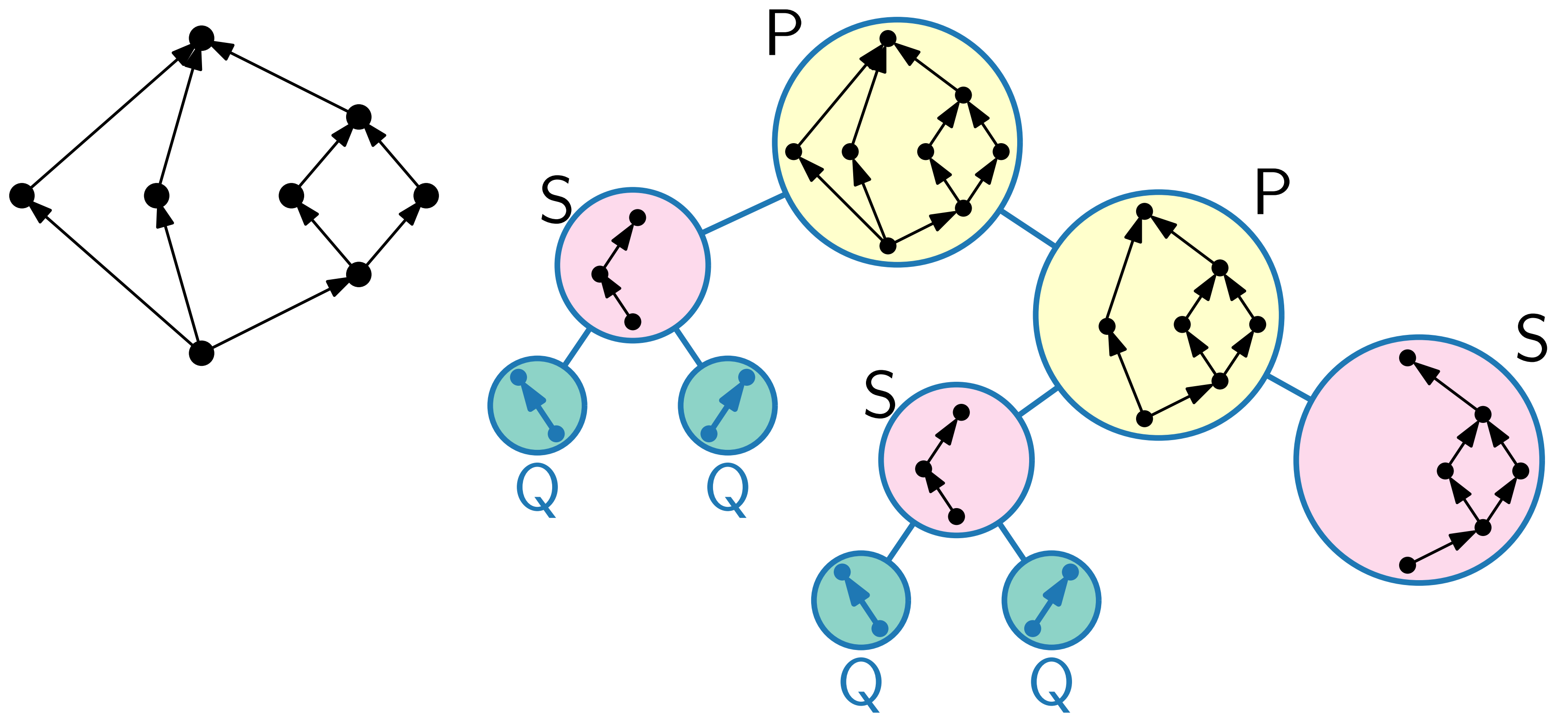

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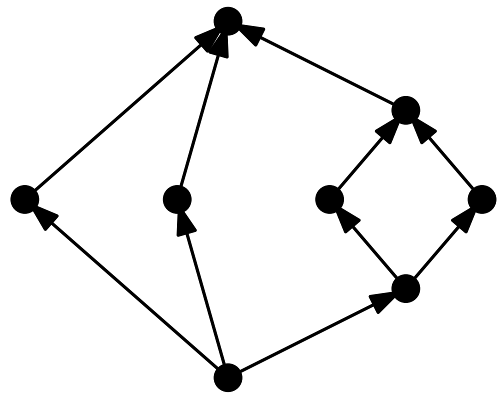

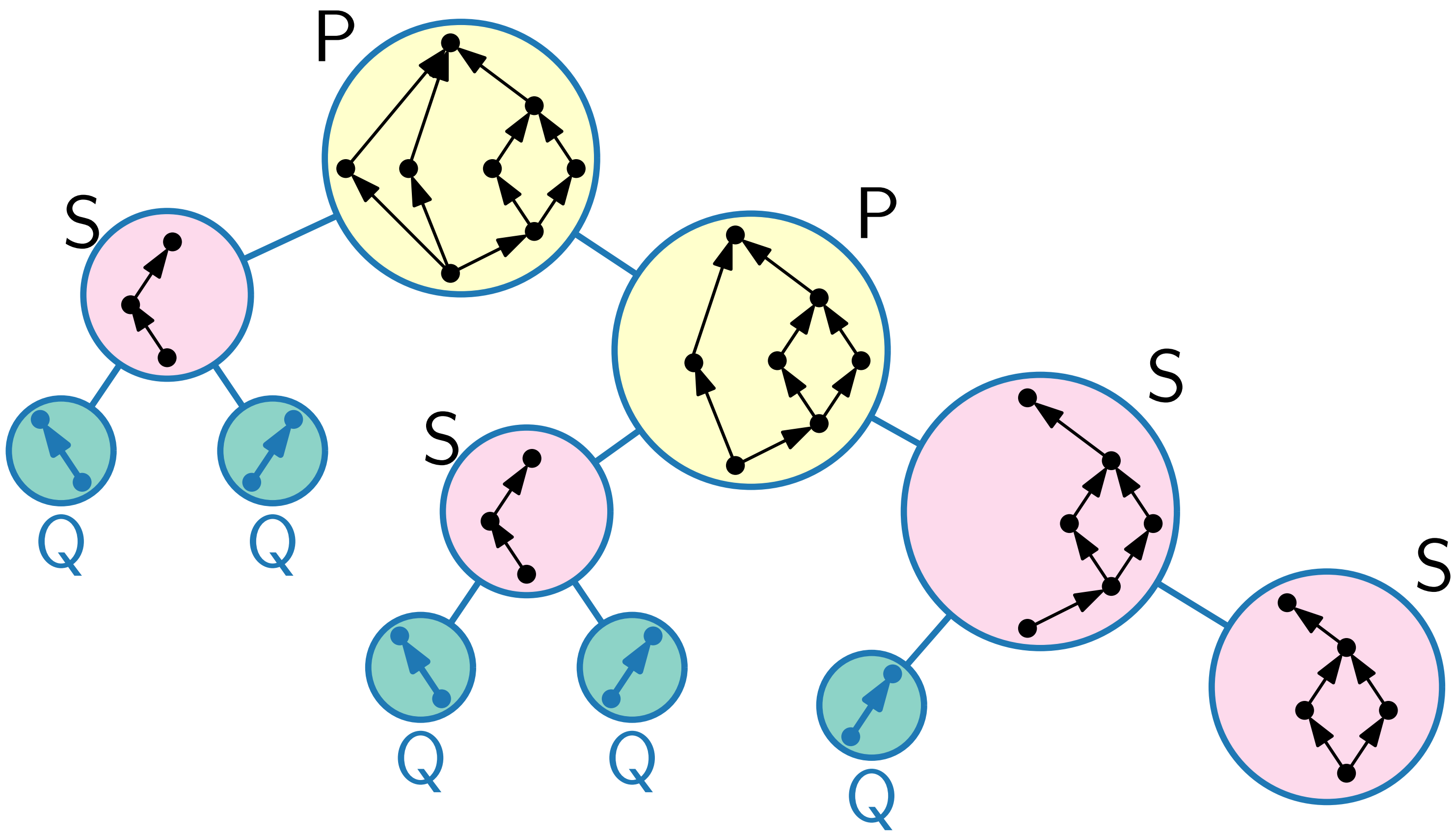

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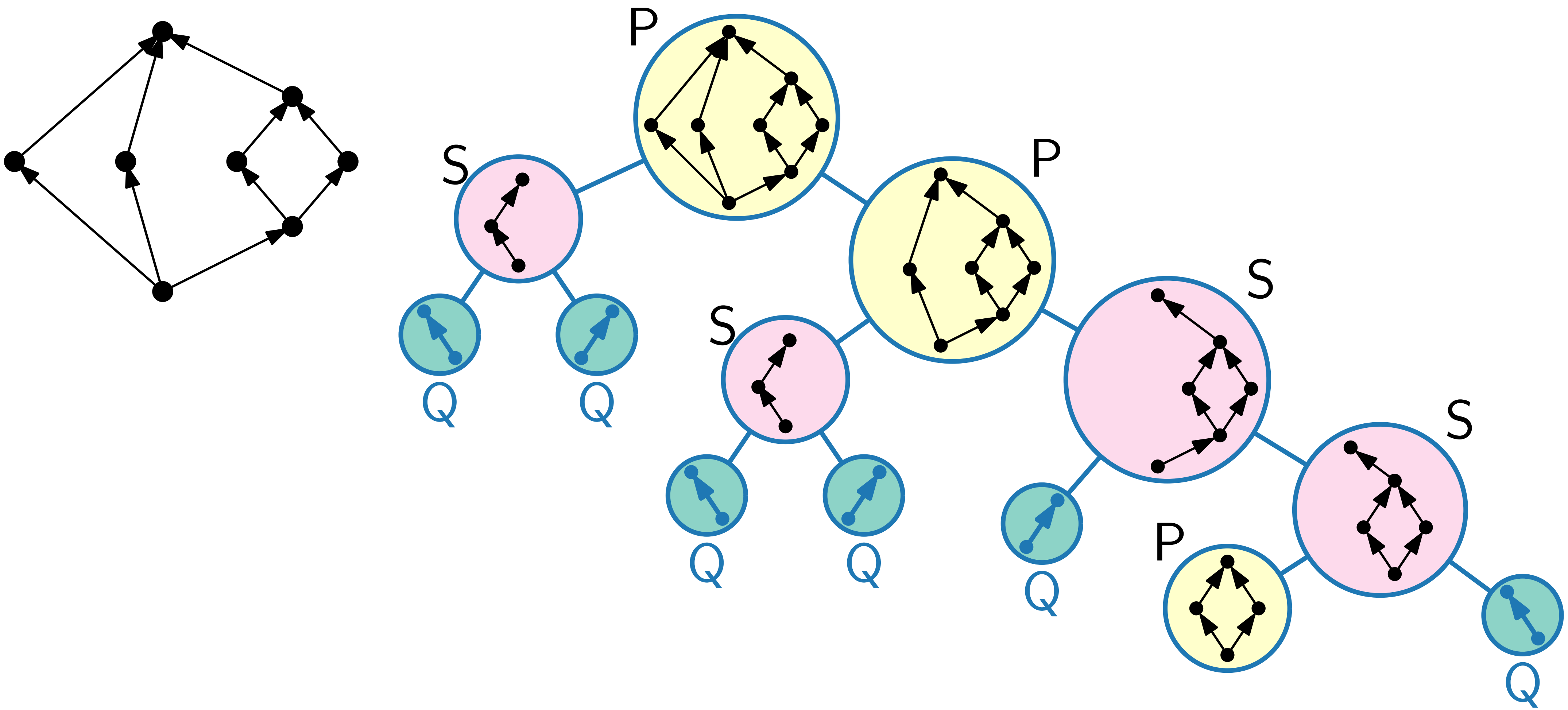

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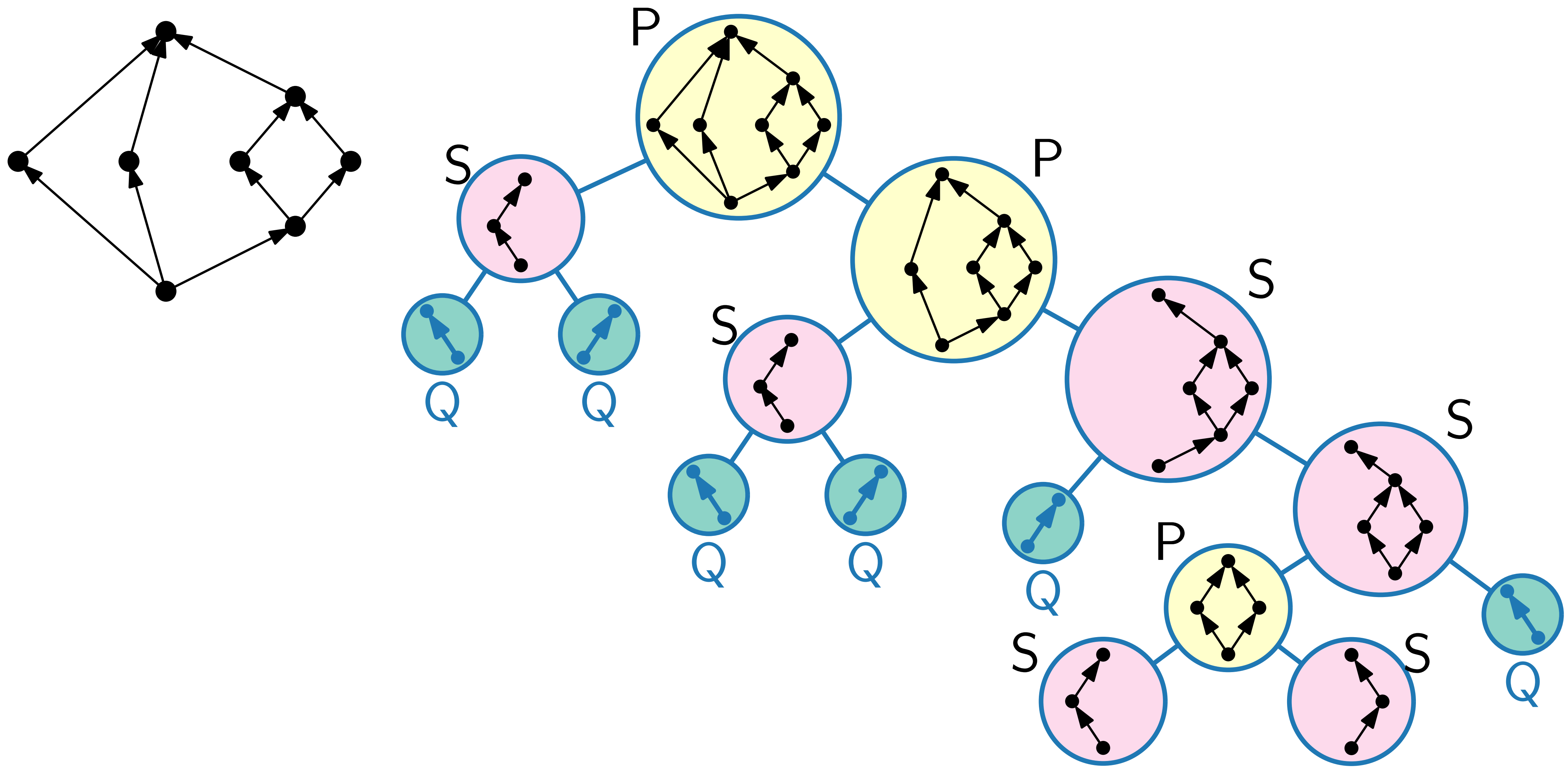

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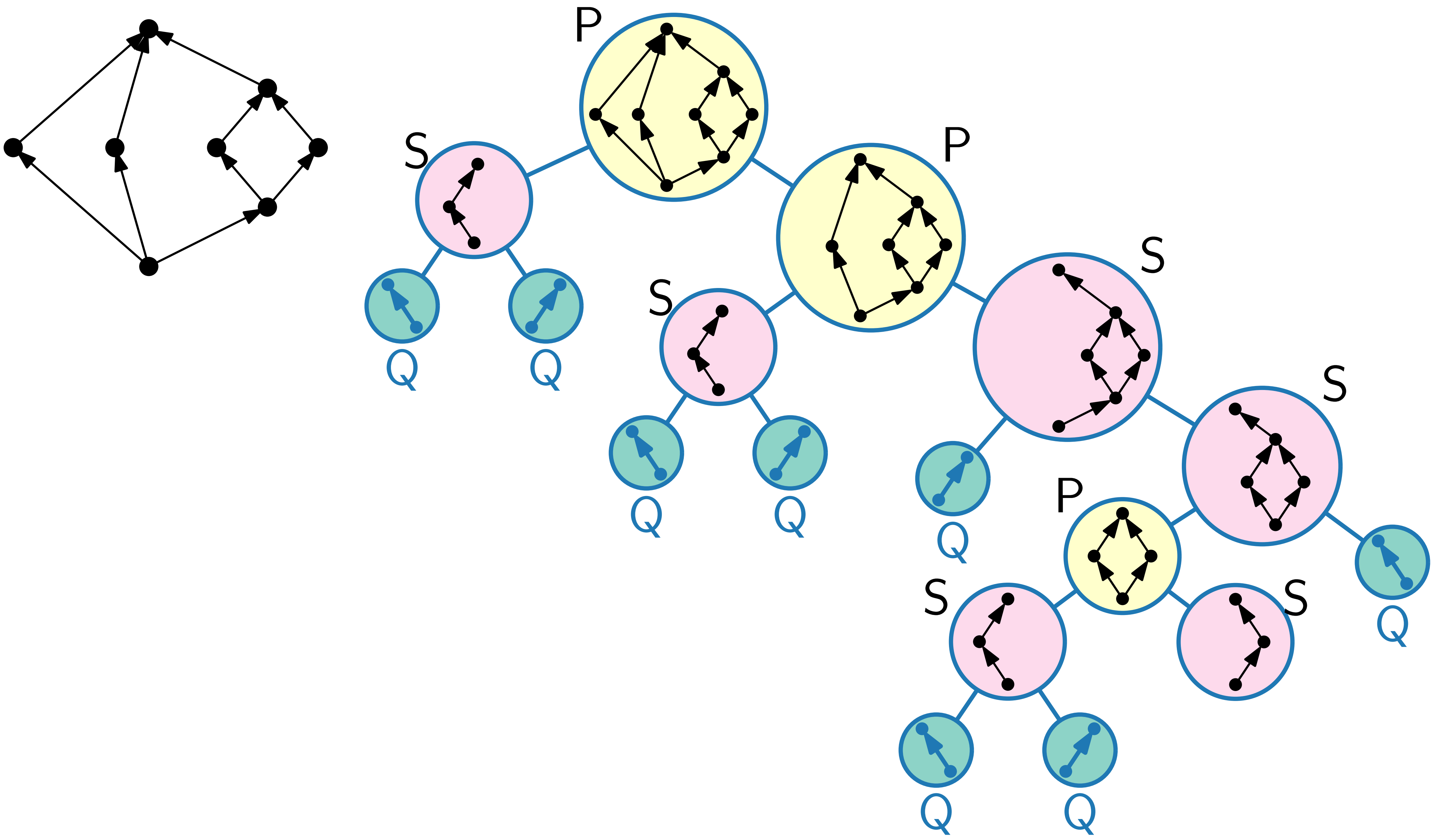

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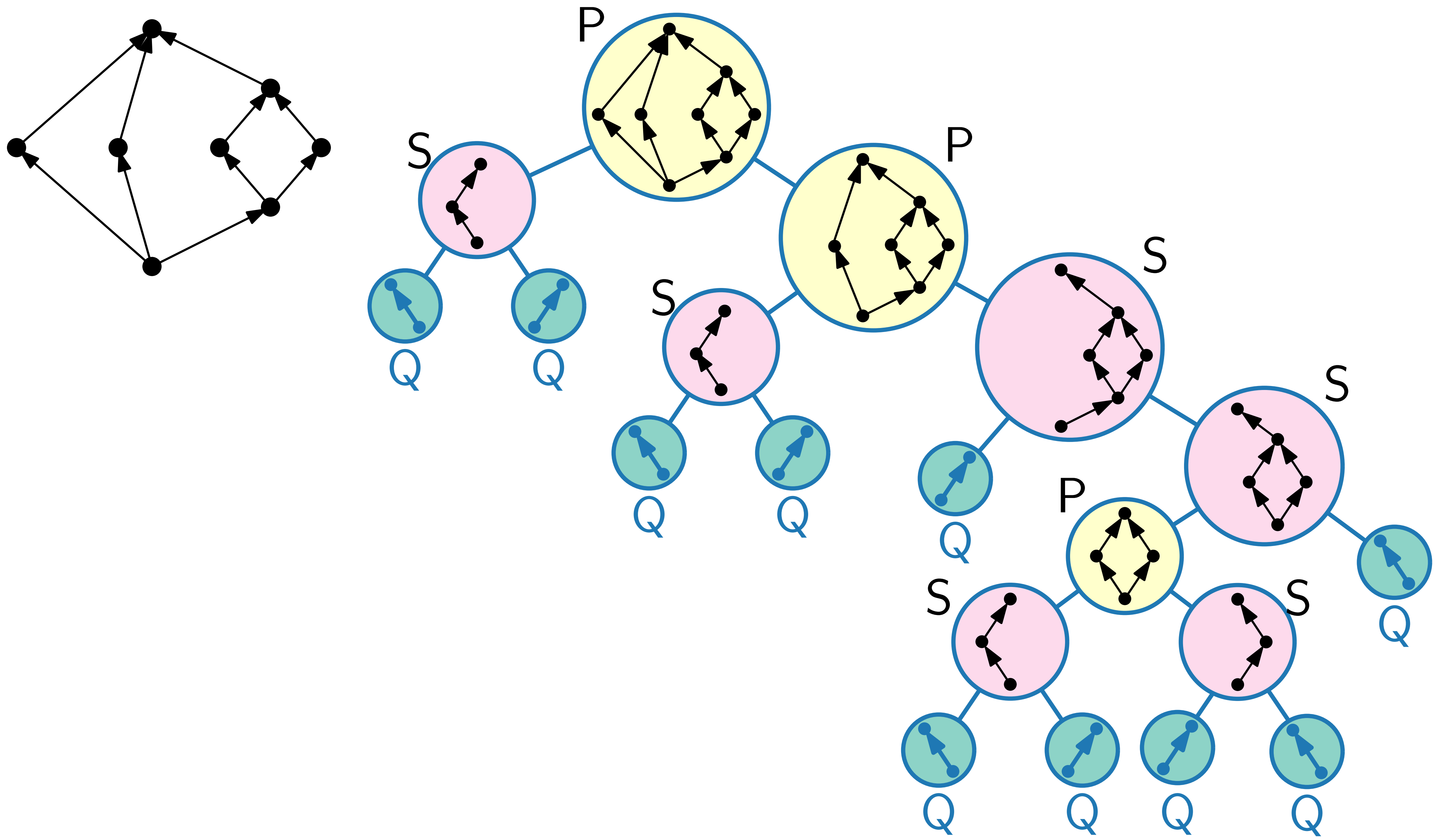
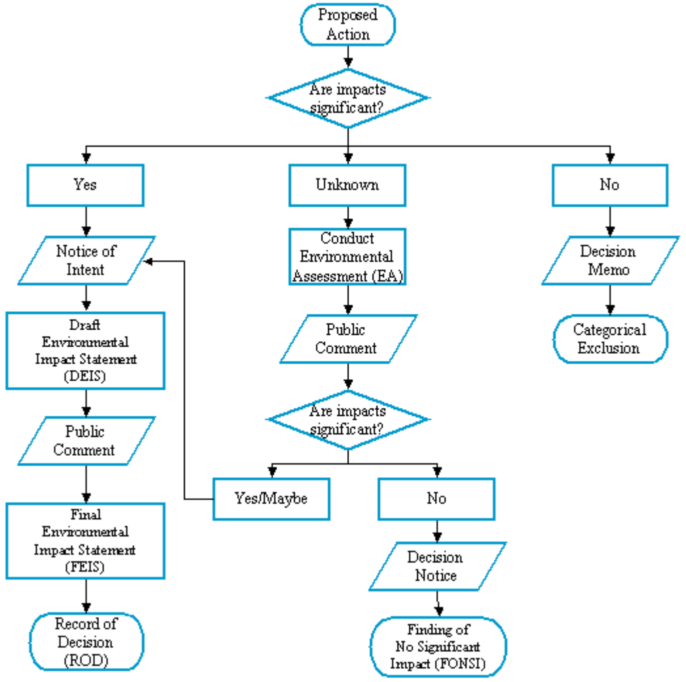
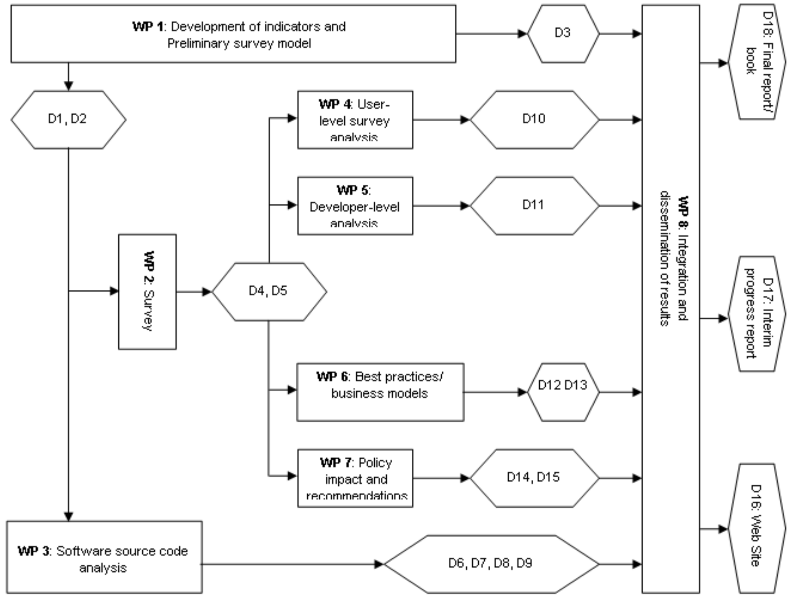


Series-parallel graphs – applications



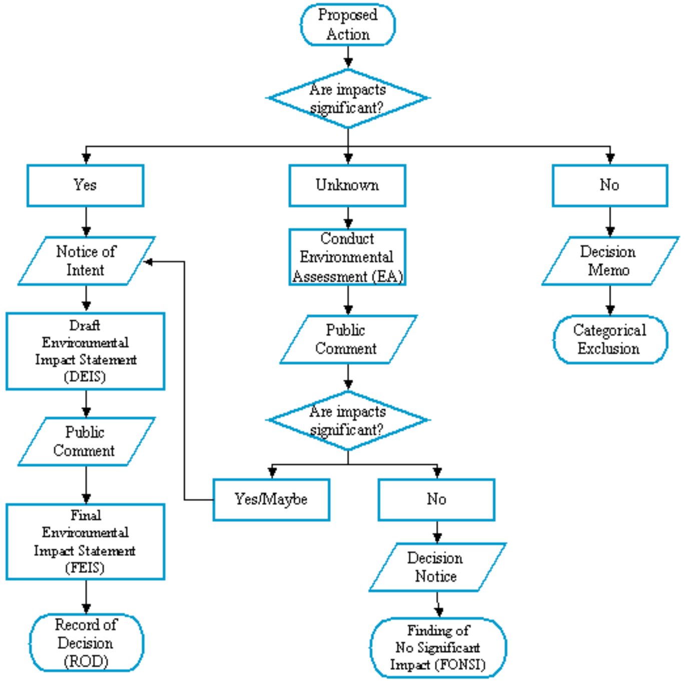
Flowcharts



PERT-Diagrams

(Program Evaluation and Review Technique)

Series-parallel graphs – applications



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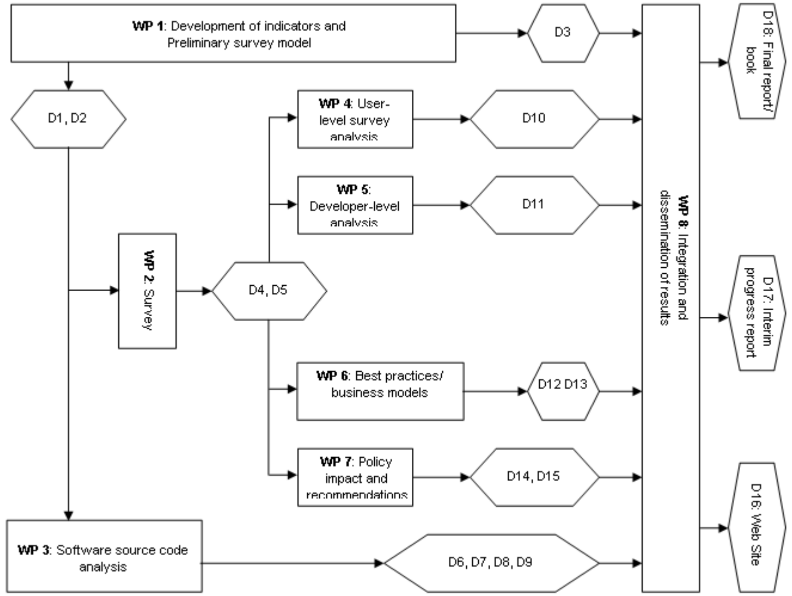


PERT-Diagrams

(Program Evaluation and Review Technique)

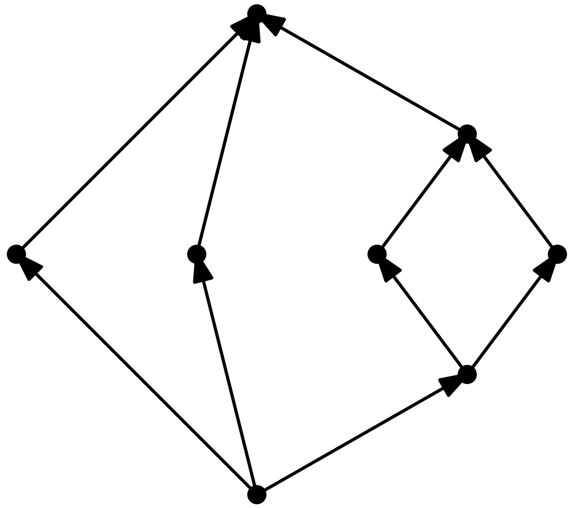
Computational complexity:

Linear time algorithms for $\mathcal{NP}$-hard problems

(e.g. Maximum Matching, MIS, Hamiltonian Completion)

Series-parallel graphs – drawing style

Drawing conventions

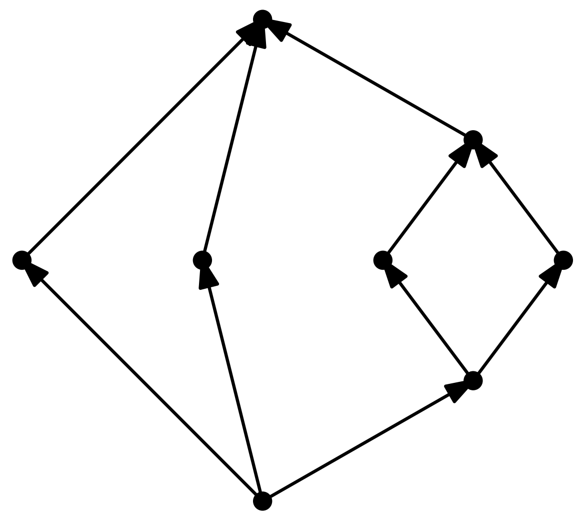


Drawing aesthetics

Series-parallel graphs – drawing style

Drawing conventions

- Planarity
- Straight-line edges
- Upward



Drawing aesthetics

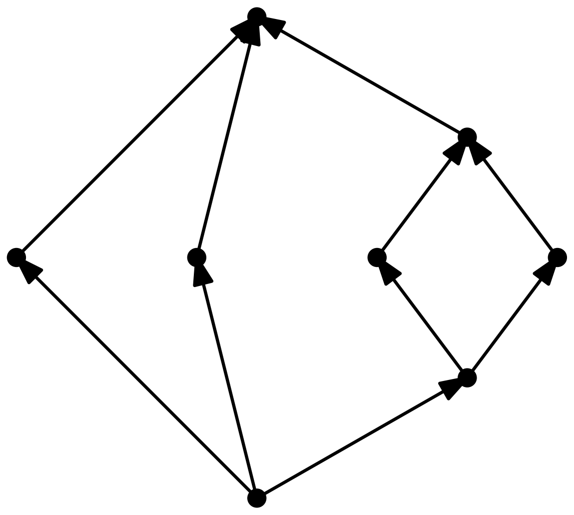
Series-parallel graphs – drawing style

Drawing conventions

- Planarity
- Straight-line edges
- Upward

Drawing aesthetics

- Area
- Symmetry



Series-parallel graphs – An exponential area bound

- **A class of graphs that requires exponential area for its upward drawing**

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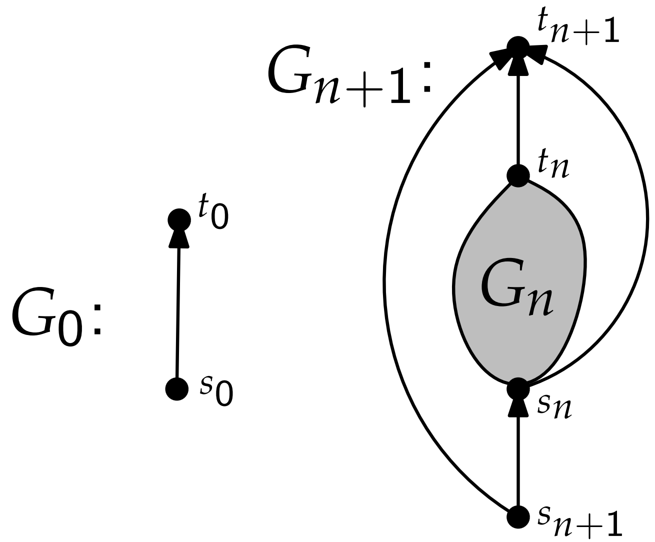

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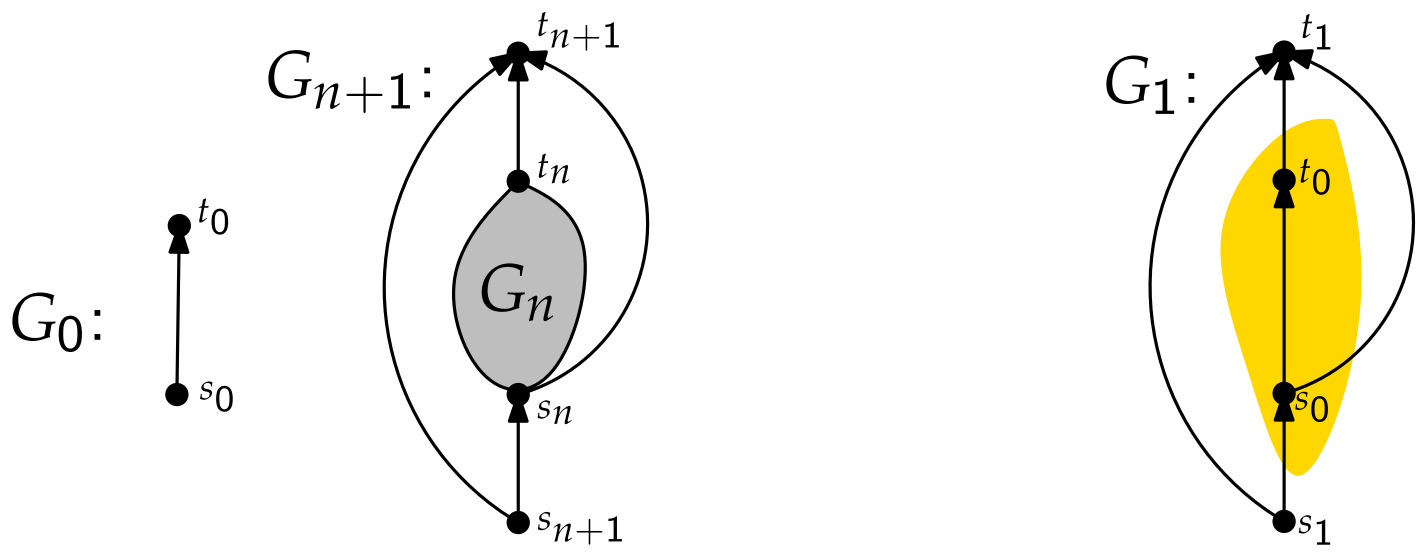

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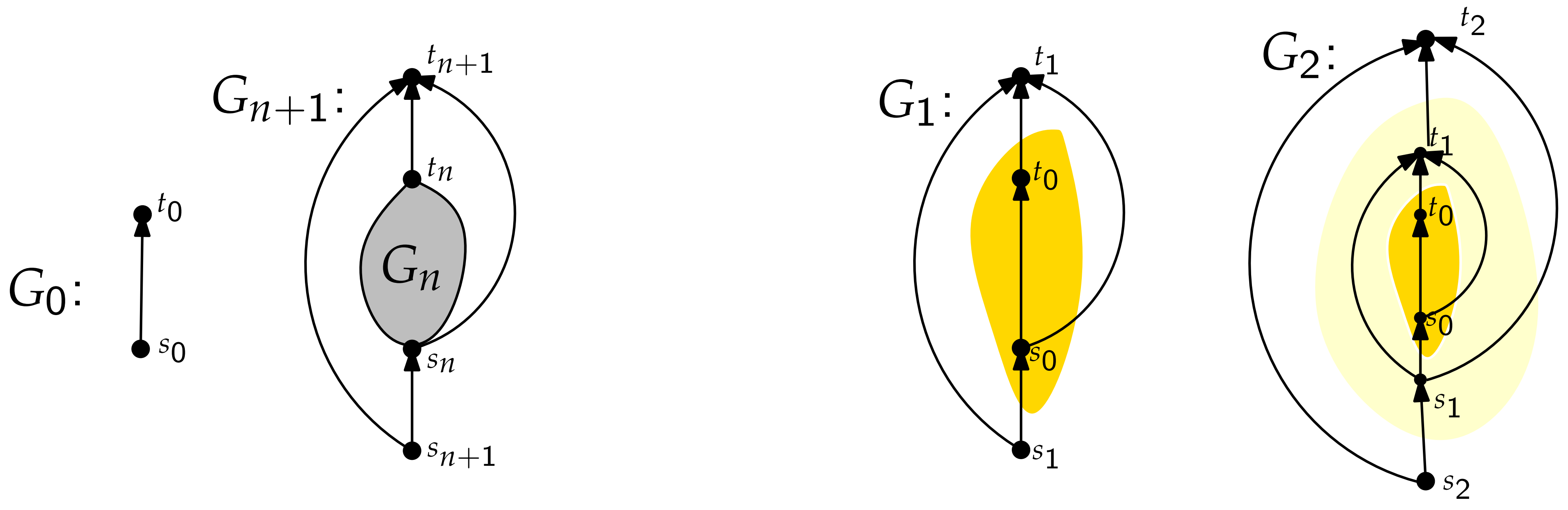

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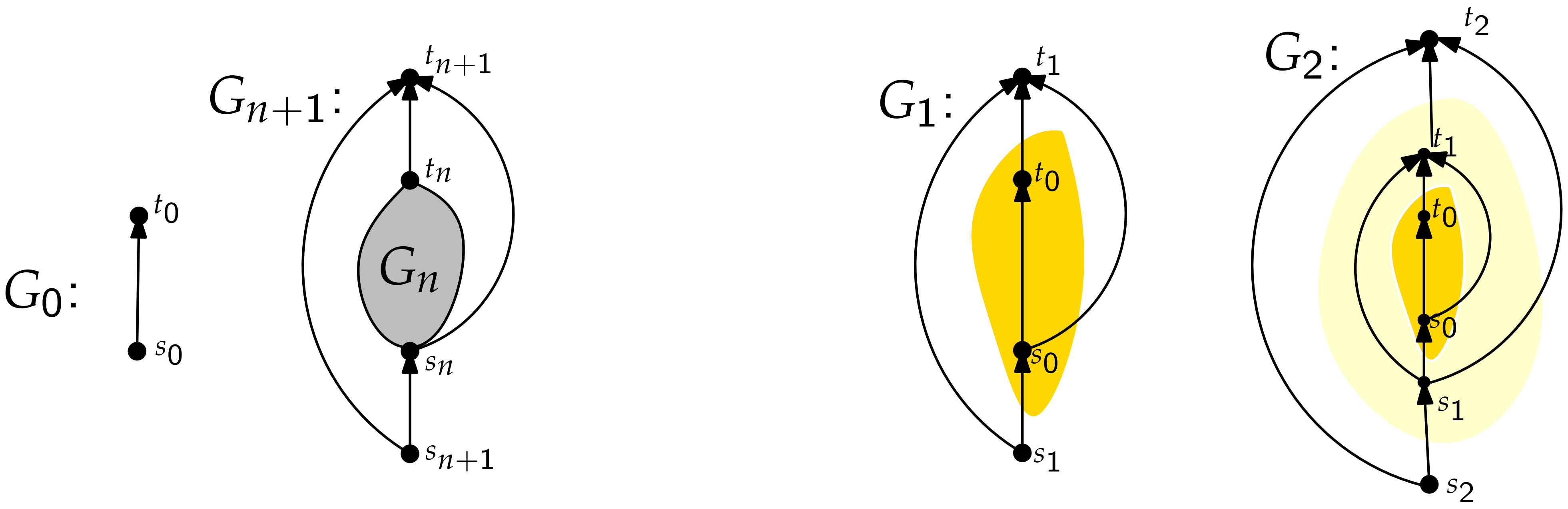

Theorem [Bertolazzi et al. 1994] Any upward drawing of the $2n$-vertex embedded graph G_n that preserves the embedding requires area $\Omega(4^n)$, under any resolution rule.

Series-parallel graphs – fixed embedding

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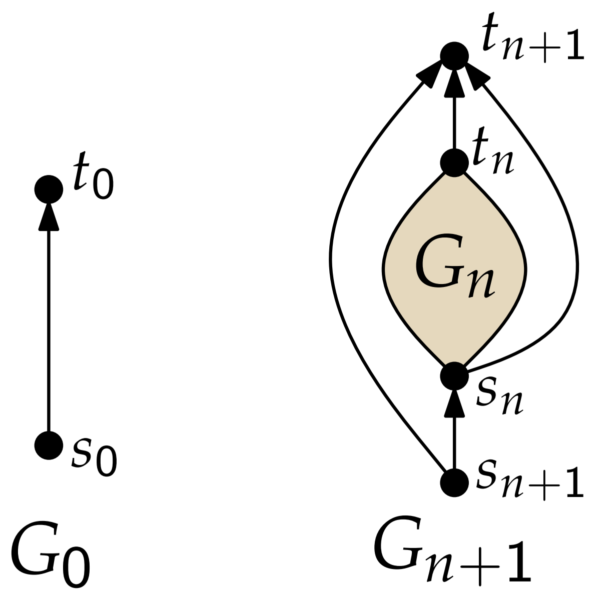
Proof:



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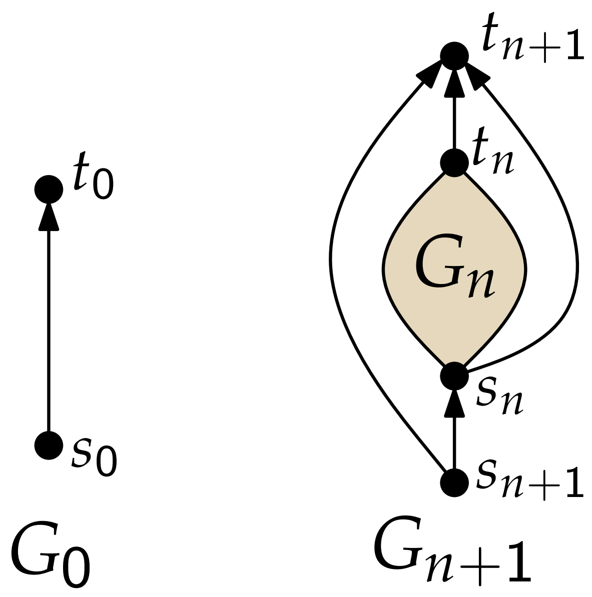

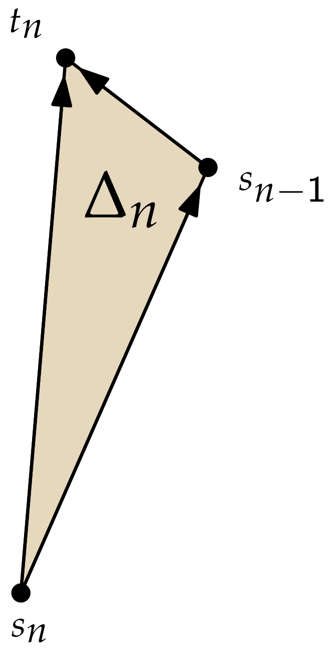

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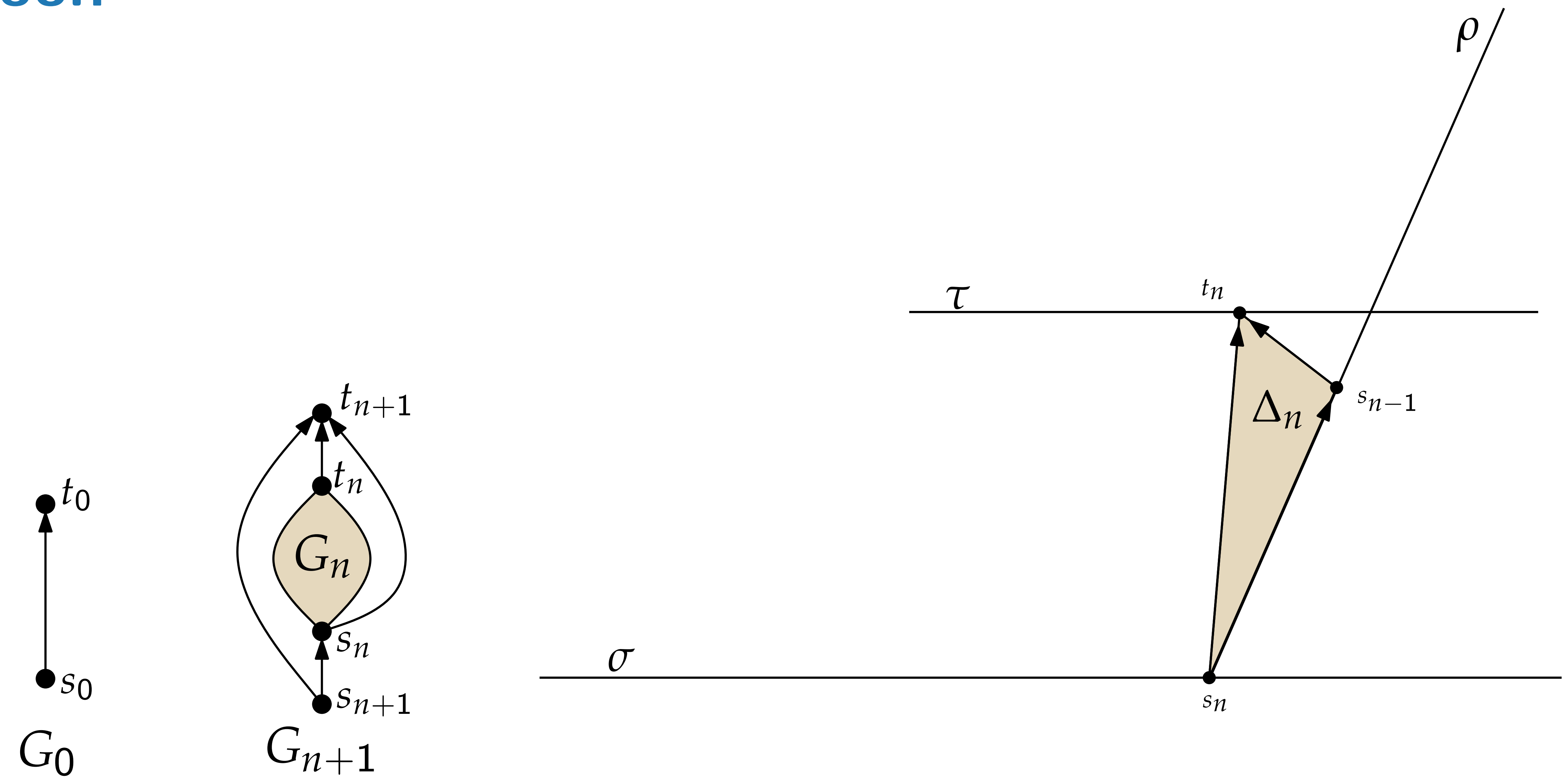
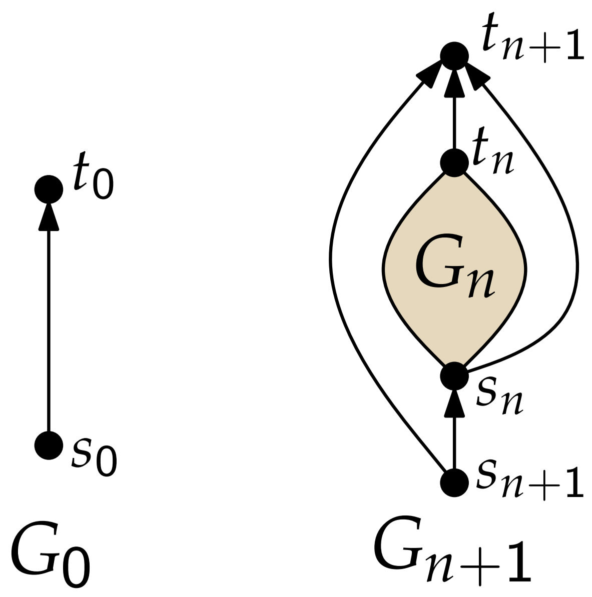
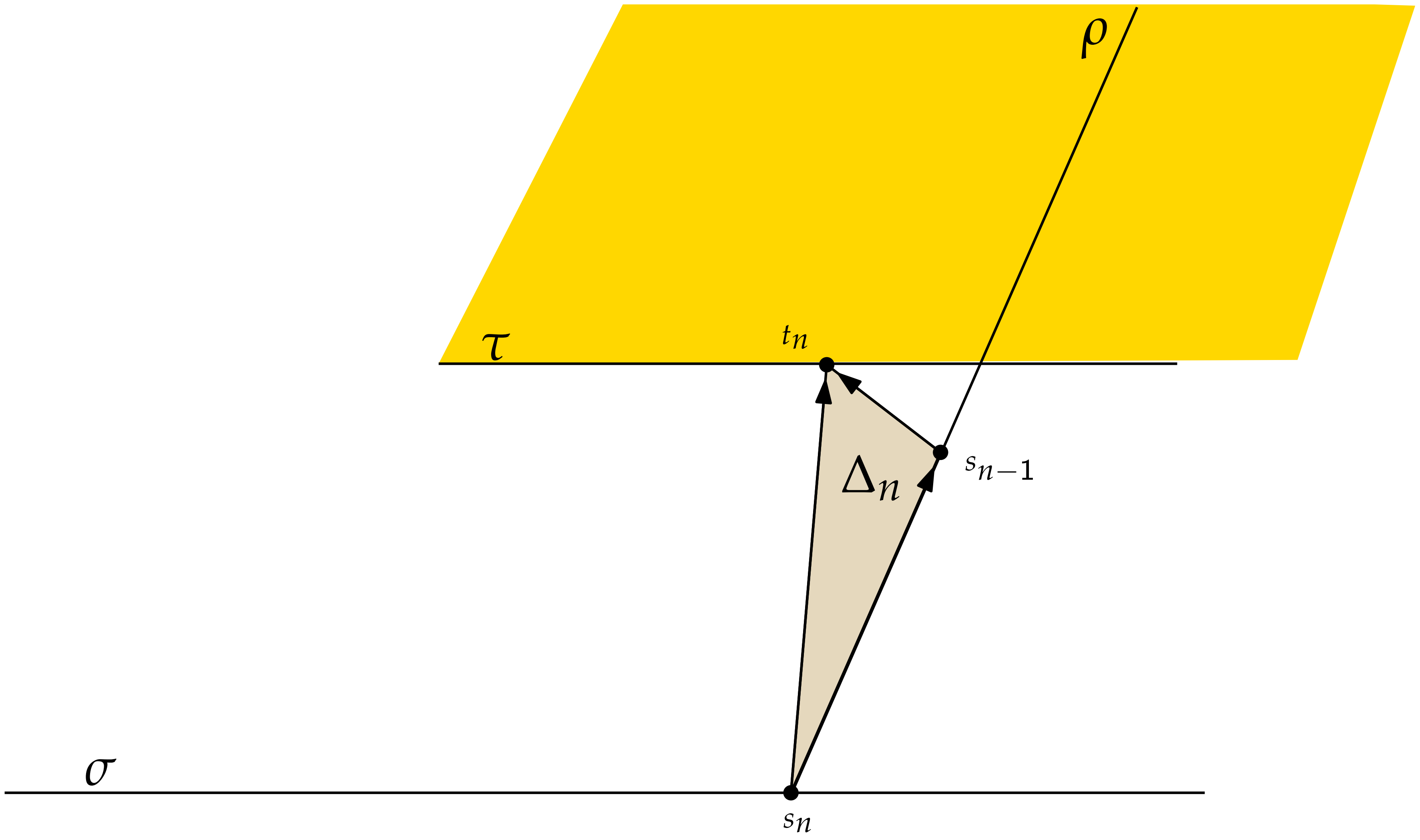


Series-parallel graphs – fixed embedding

Proof:

t_{n+1}: – above τ





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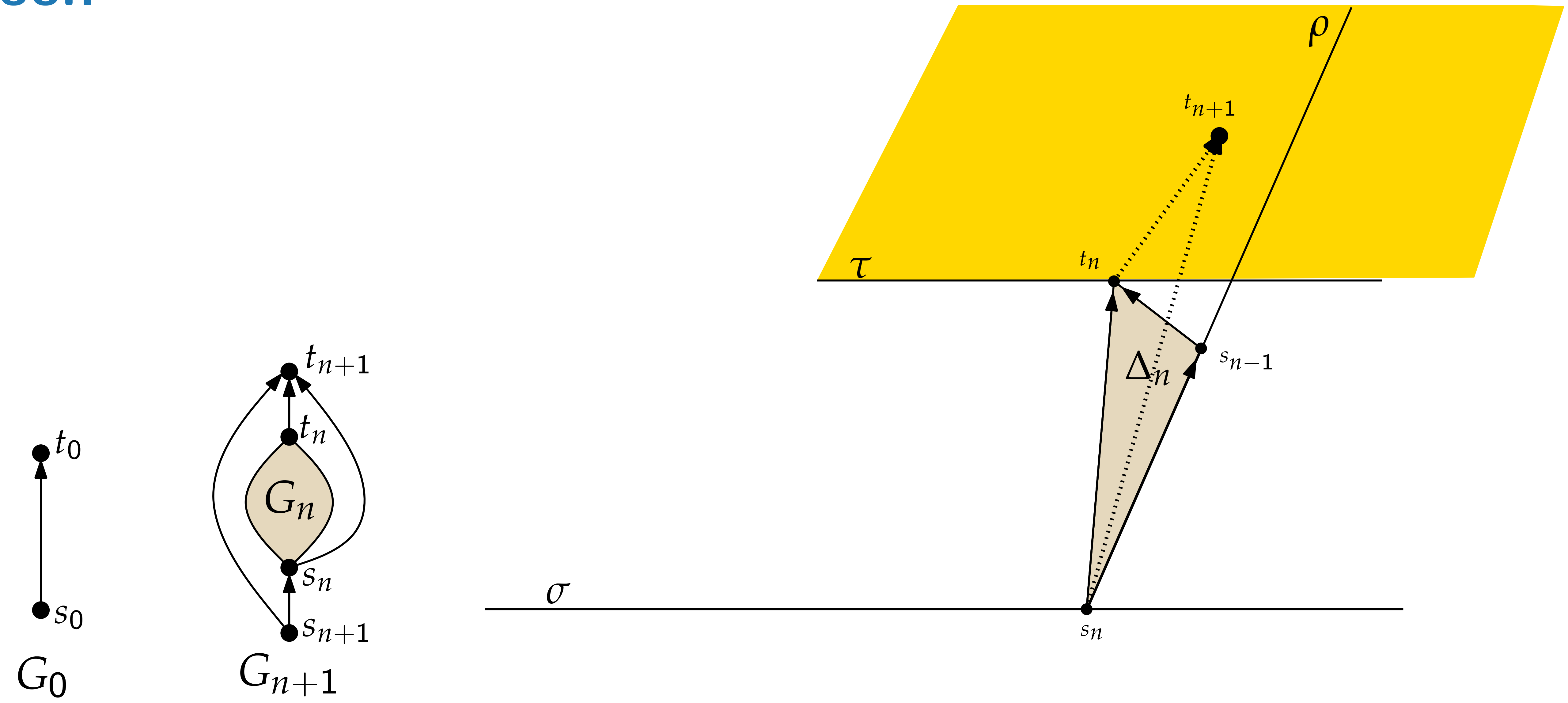

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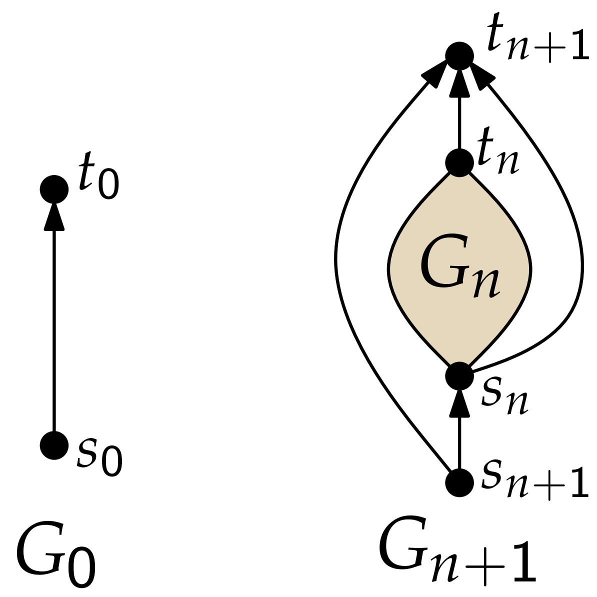
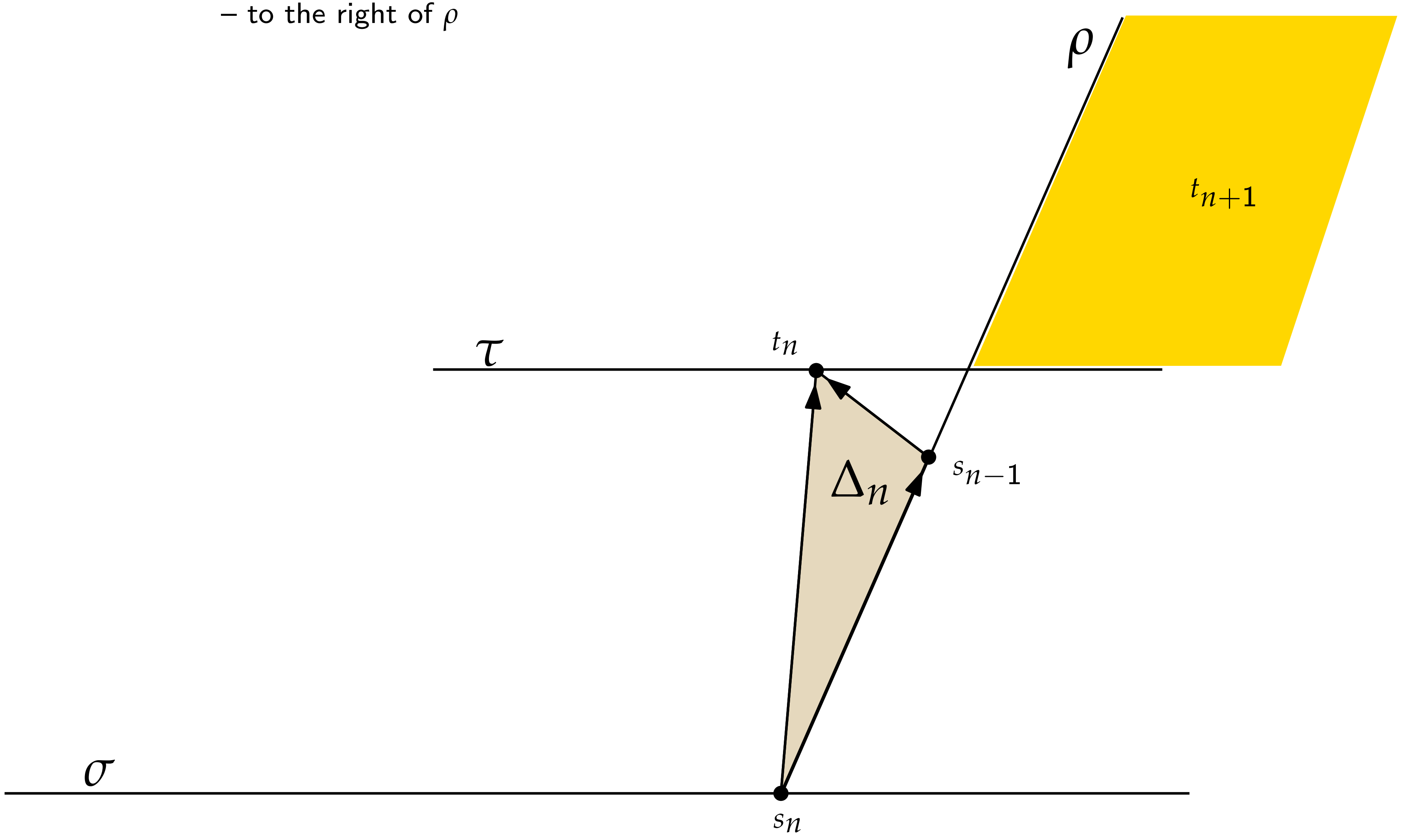
Proof:

t_{n+1}: – above τ
– to the right of ρ





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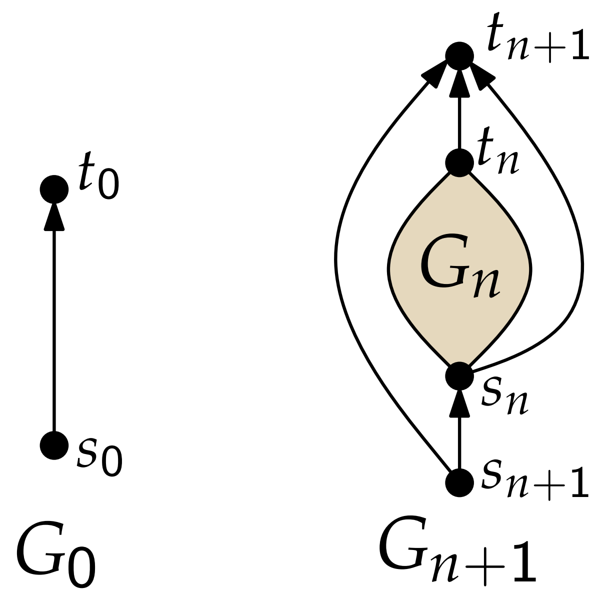
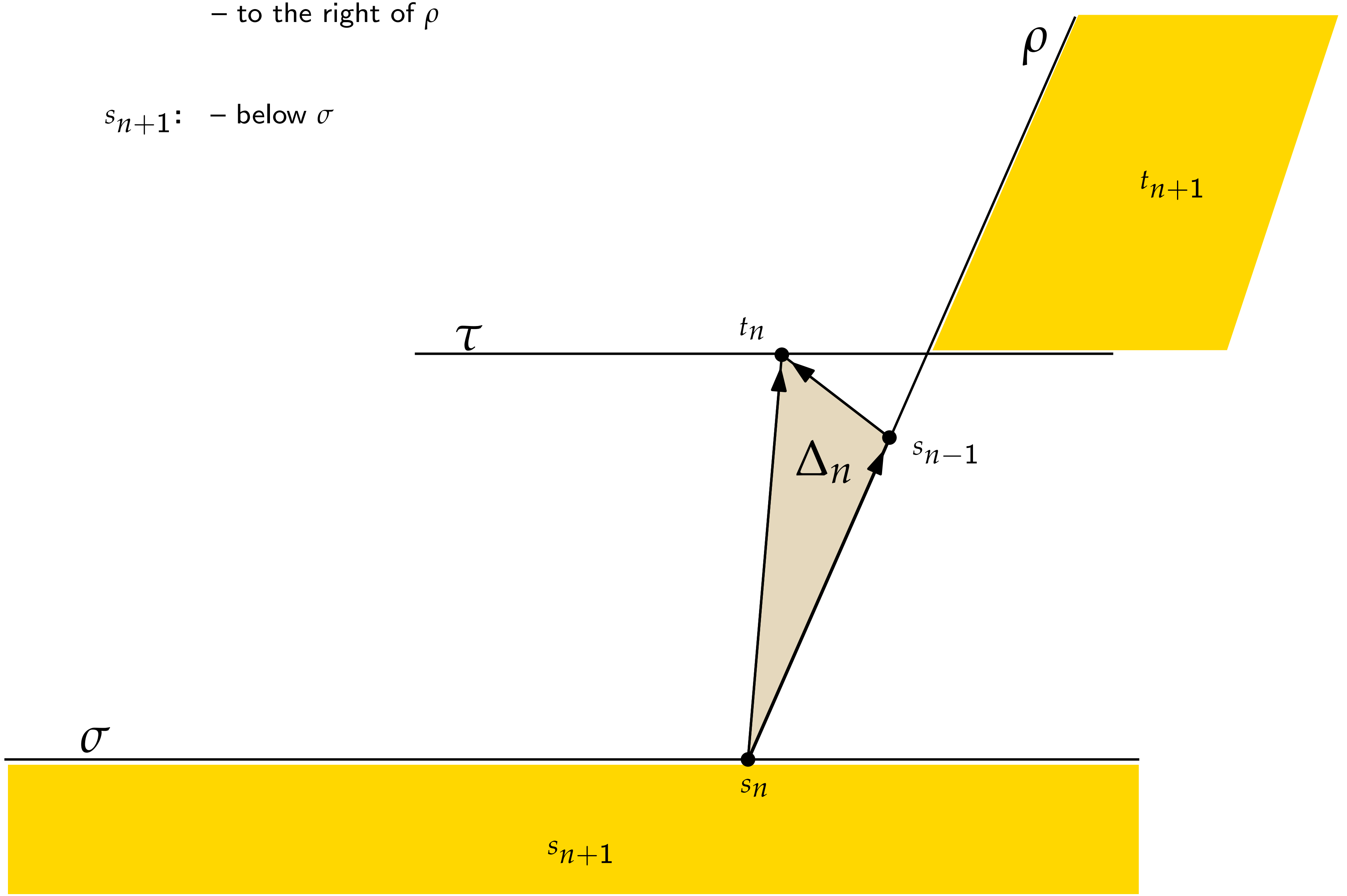
s_{n+1}: – below σ





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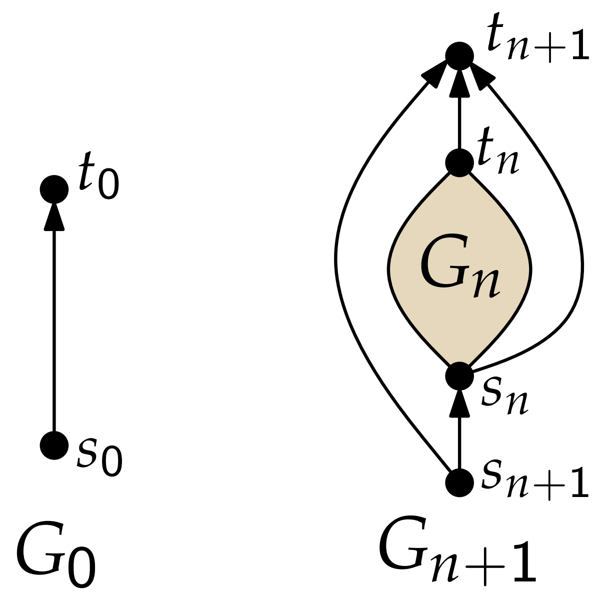

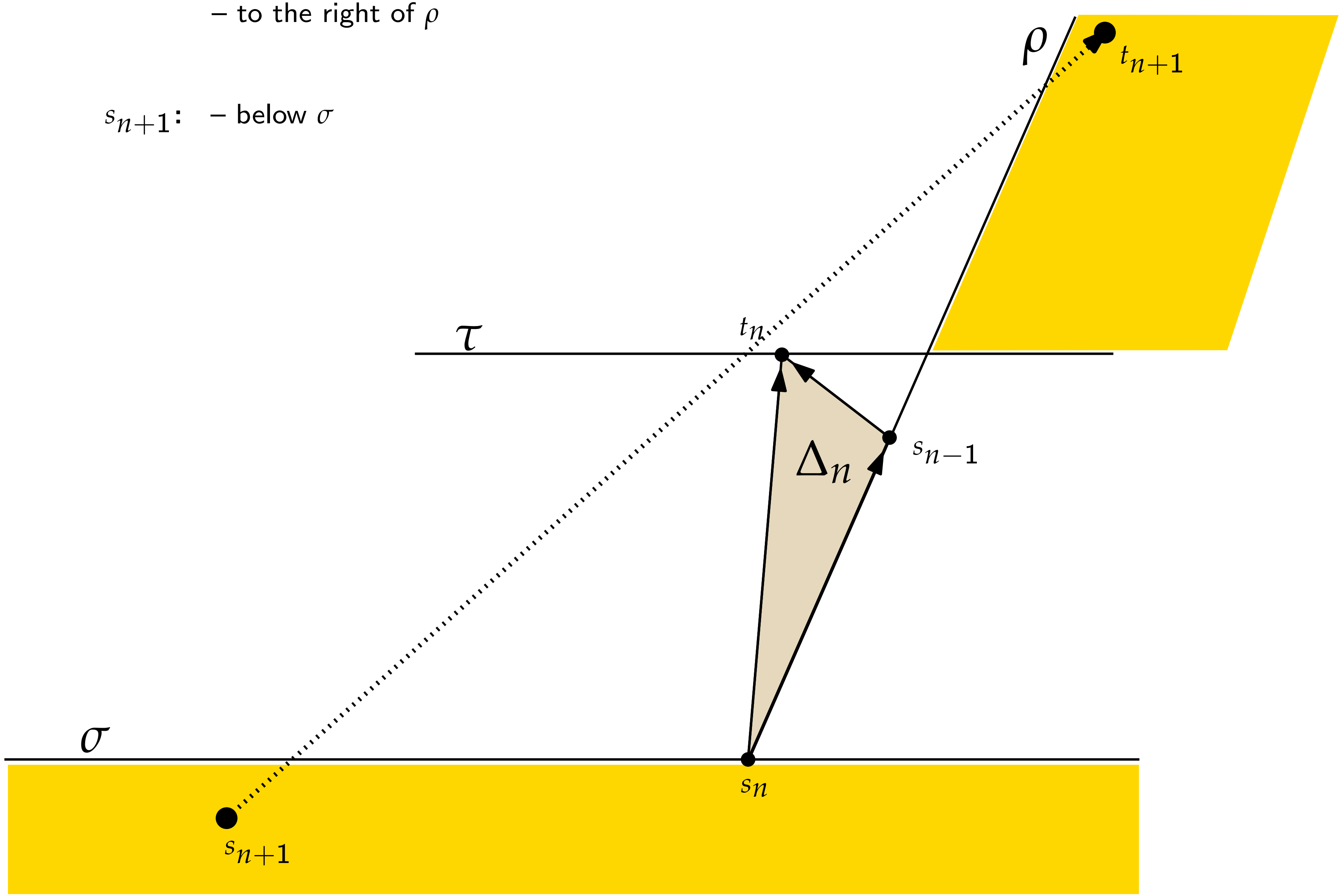

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t_{n+1}: – above τ
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– to the left of λ

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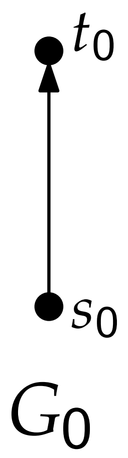

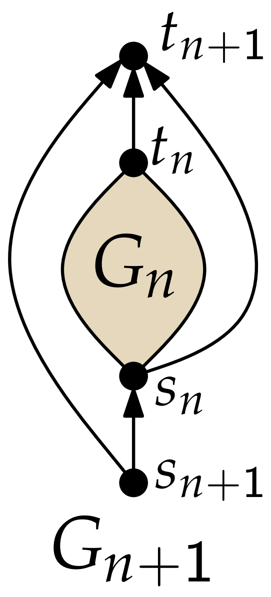

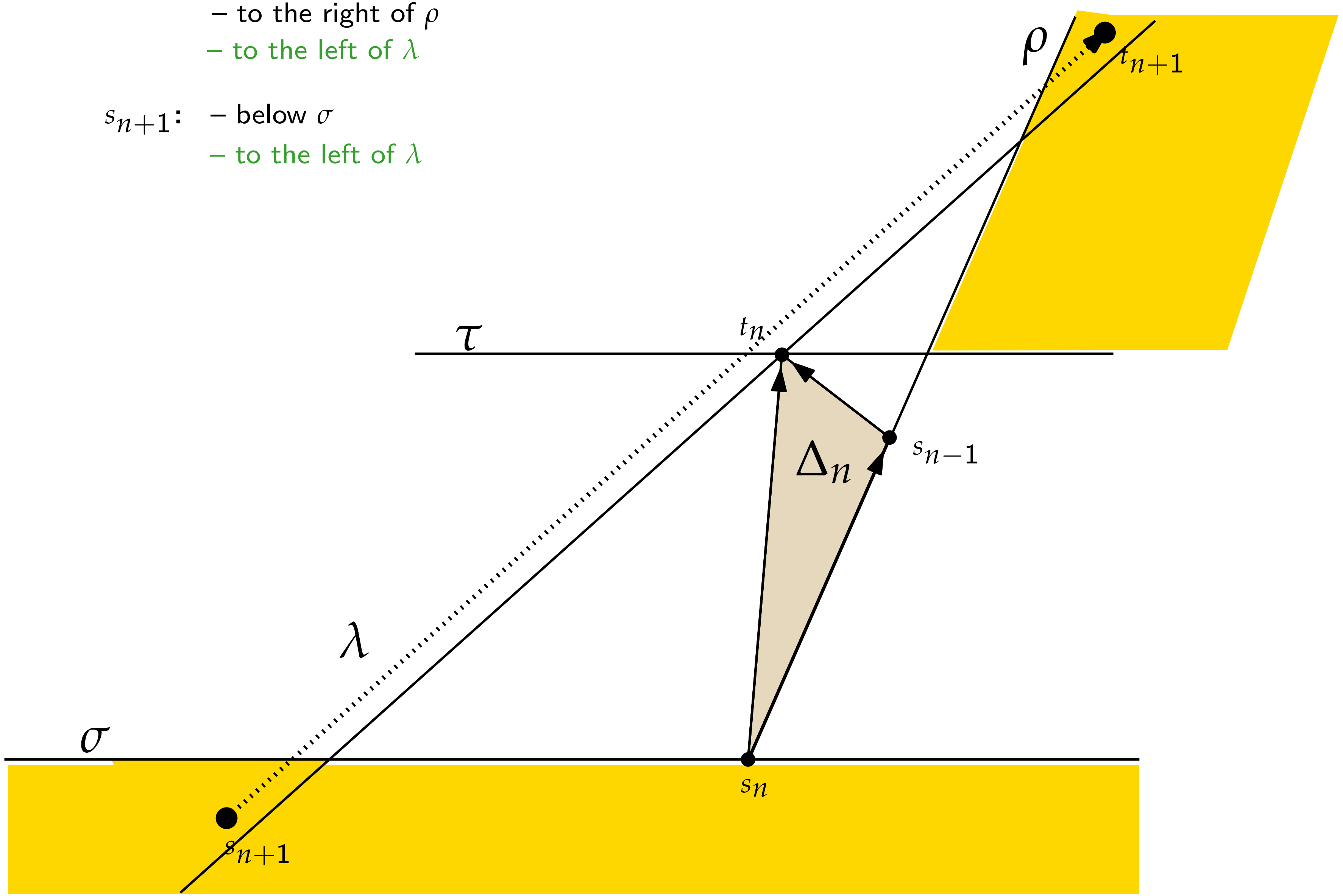

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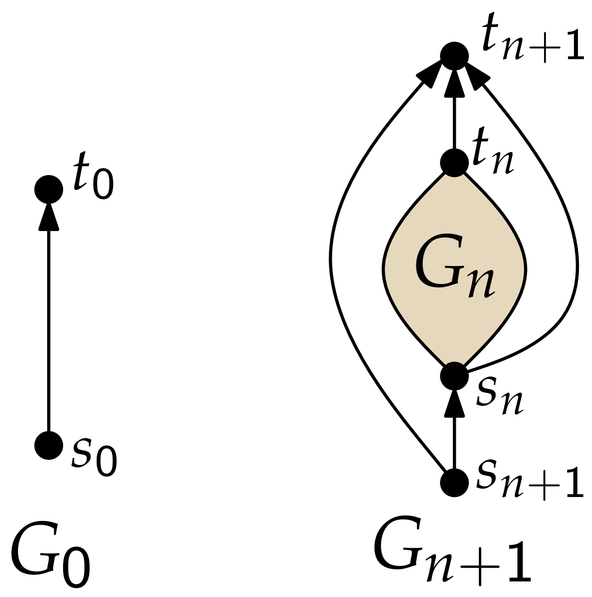

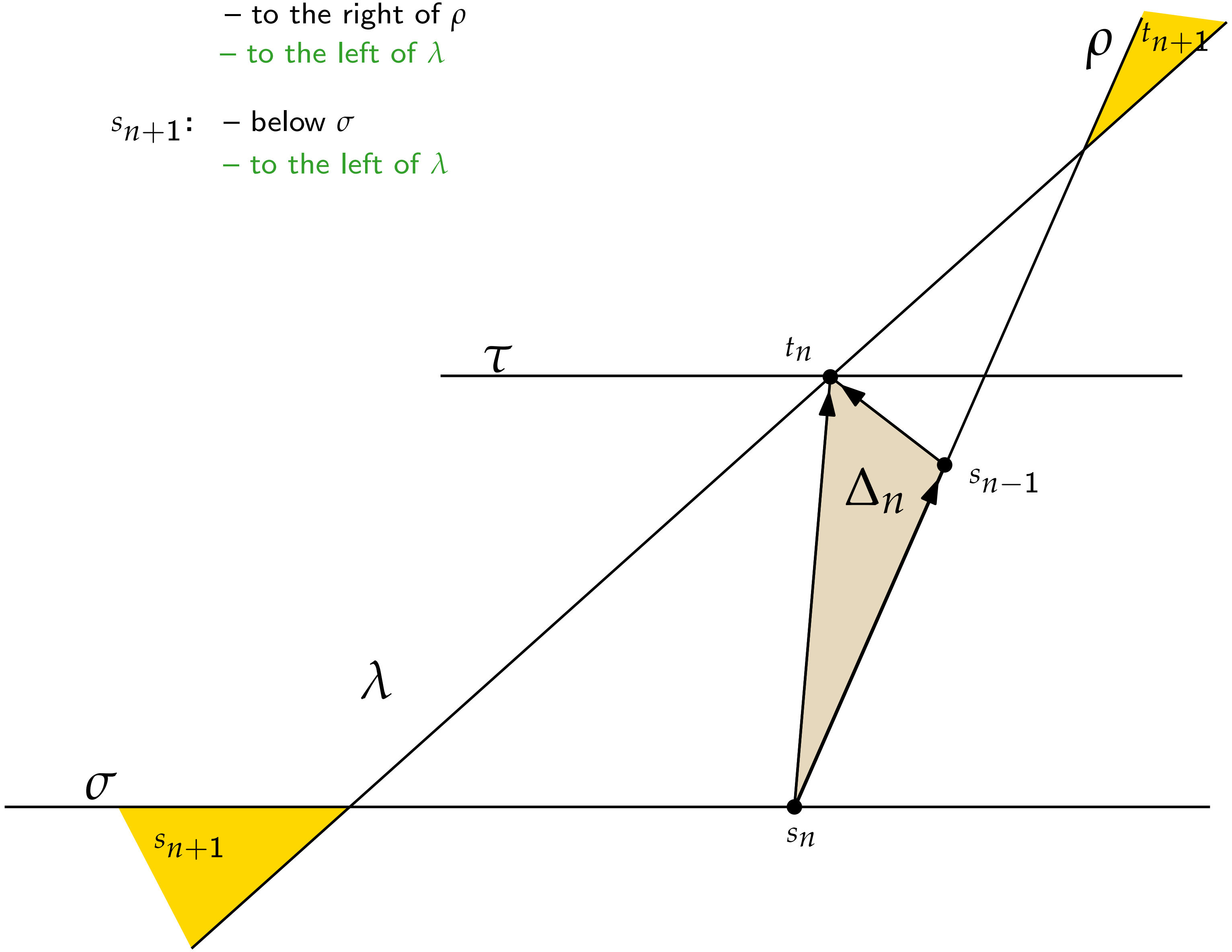

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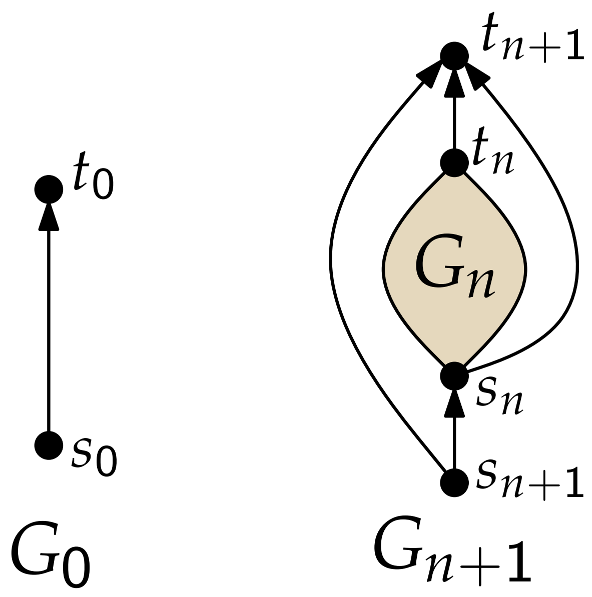
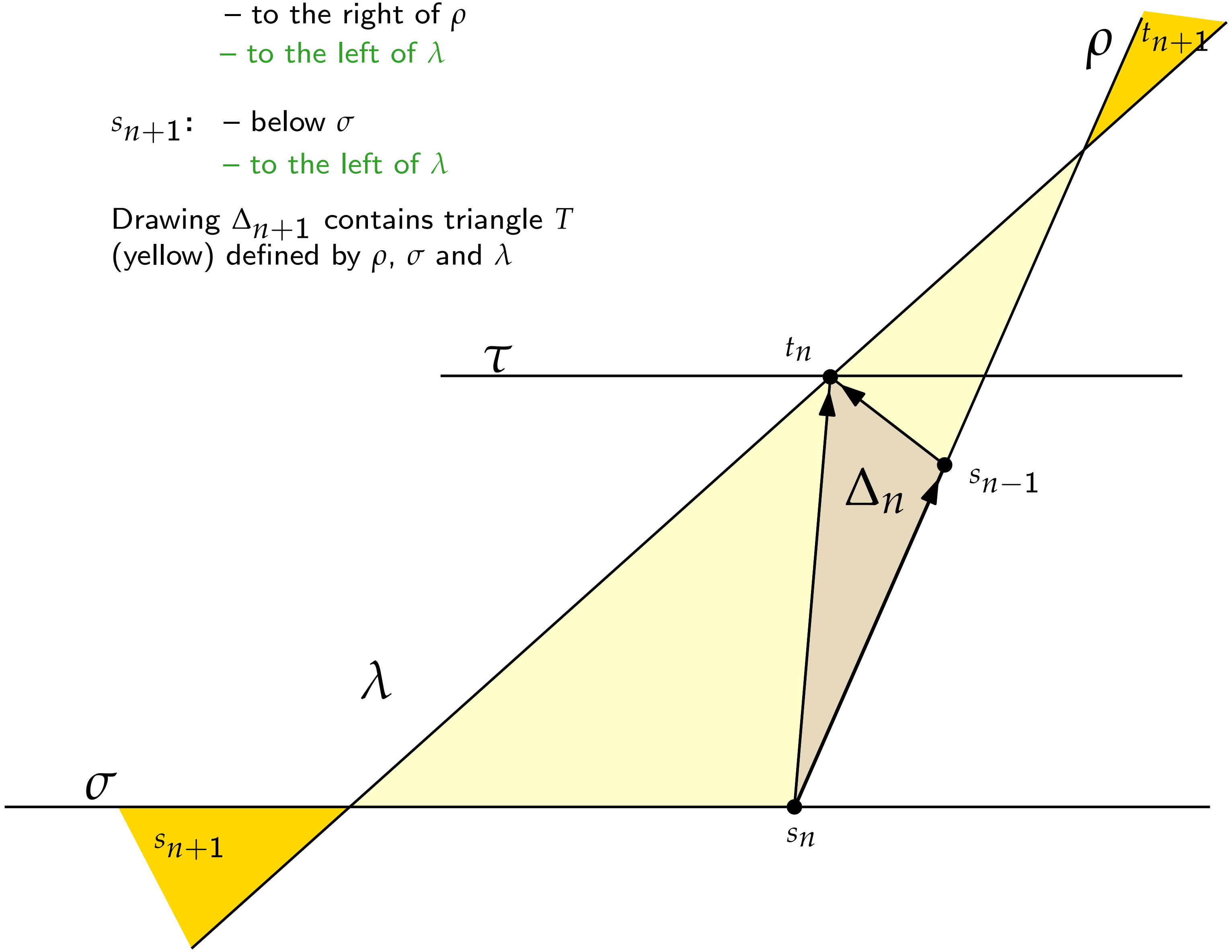
Drawing Δ_{n+1} contains triangle T (yellow) defined by ρ, σ and λ





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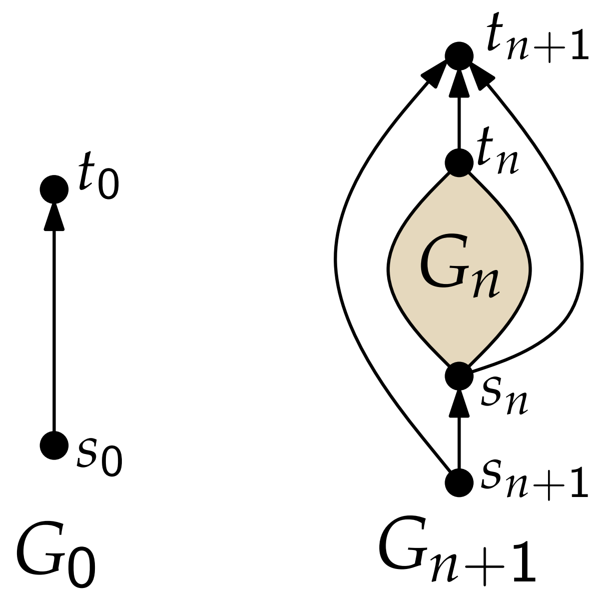
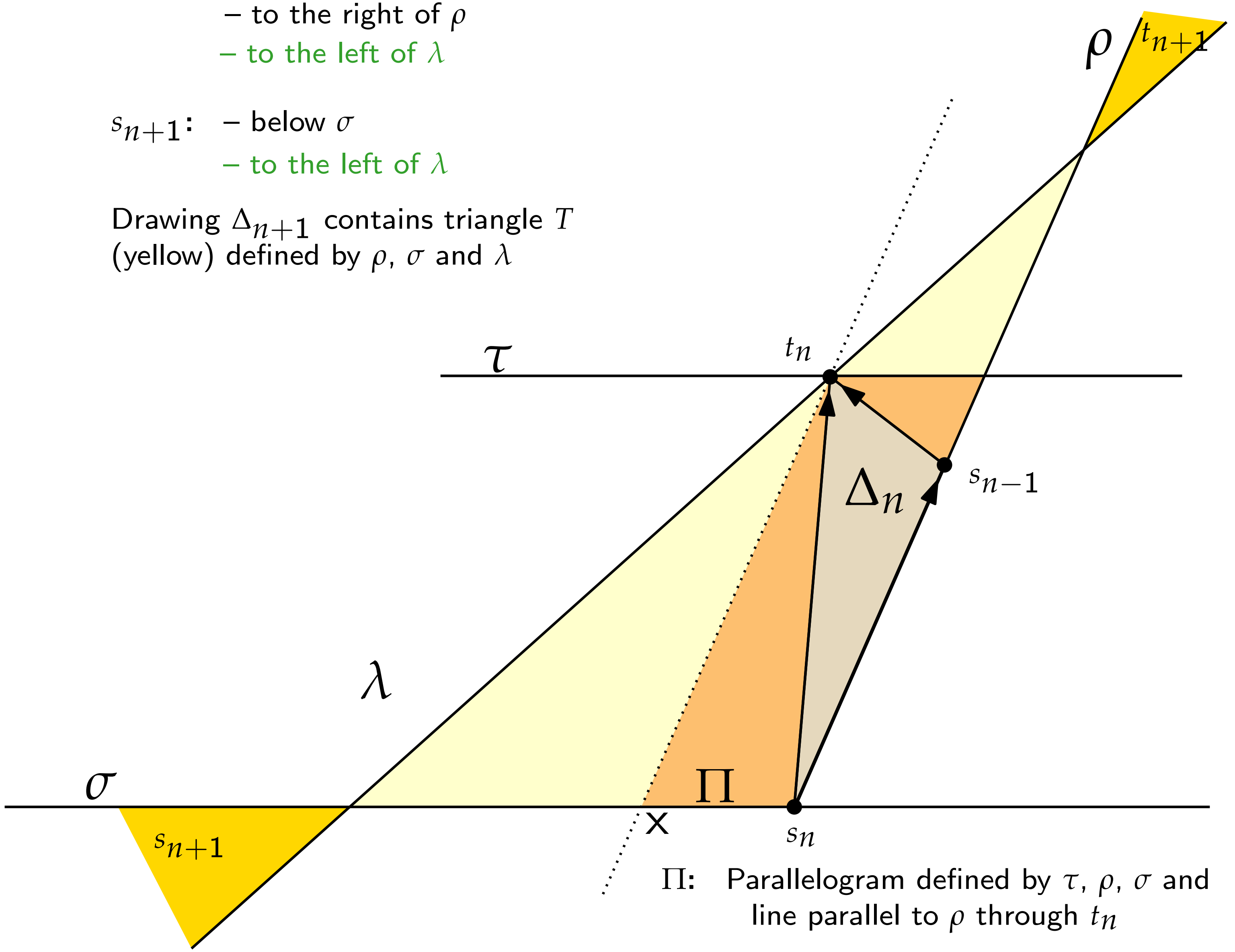


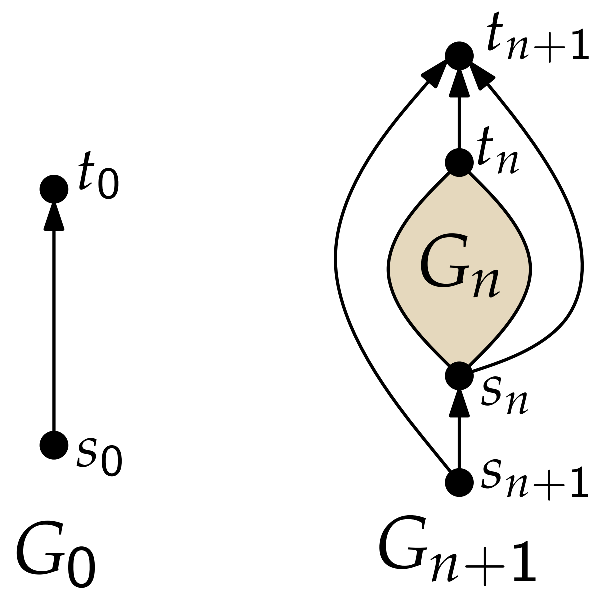


Π: Parallelogram defined by τ, ρ, σ and line parallel to ρ through t_n

Series-parallel graphs – fixed embedding

Proof:

- $2 \cdot Area(\Delta_n) < Area(\Pi)$
  [$\overline{s_n, t_n}$ is the diagonal of Π]



t_{n+1}: – above τ
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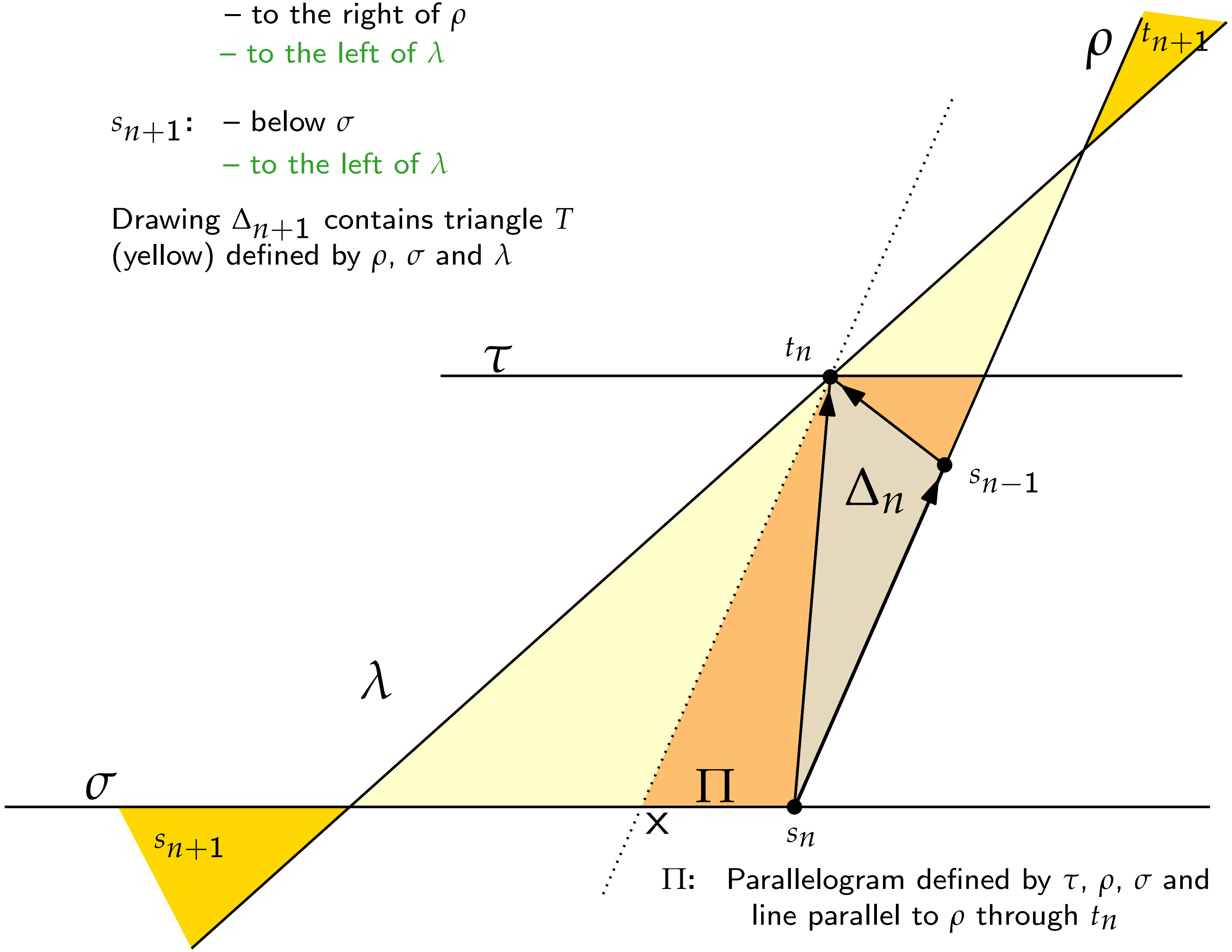

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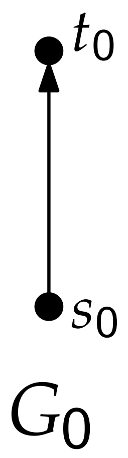

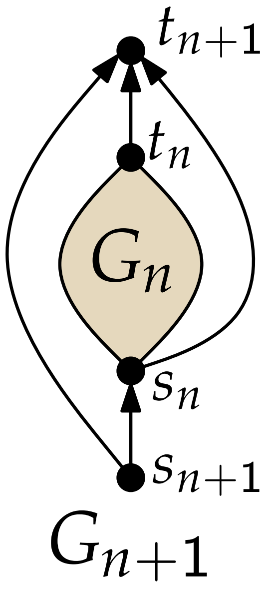

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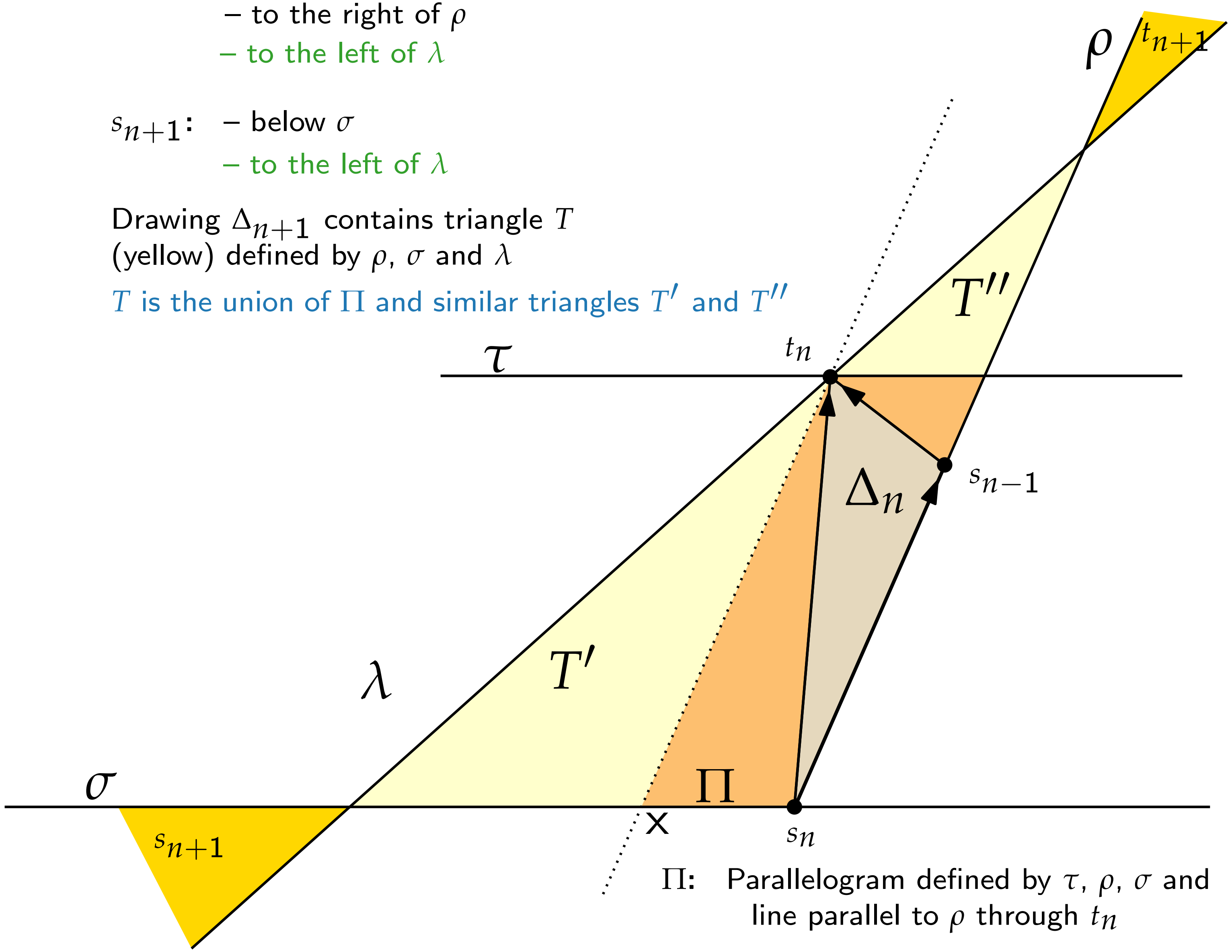

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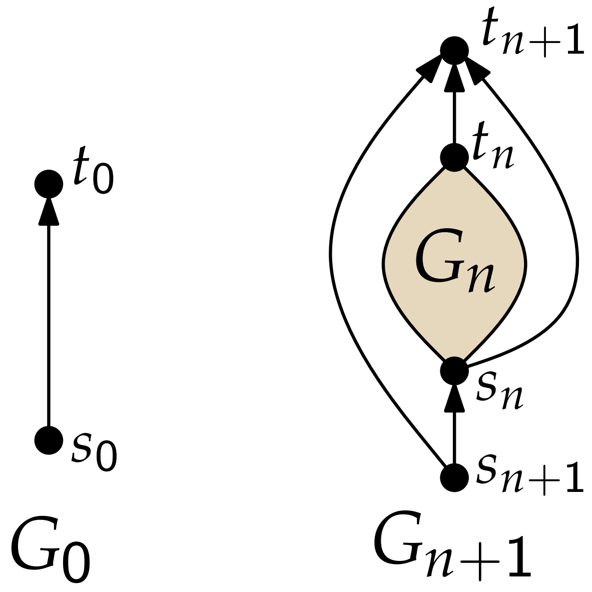

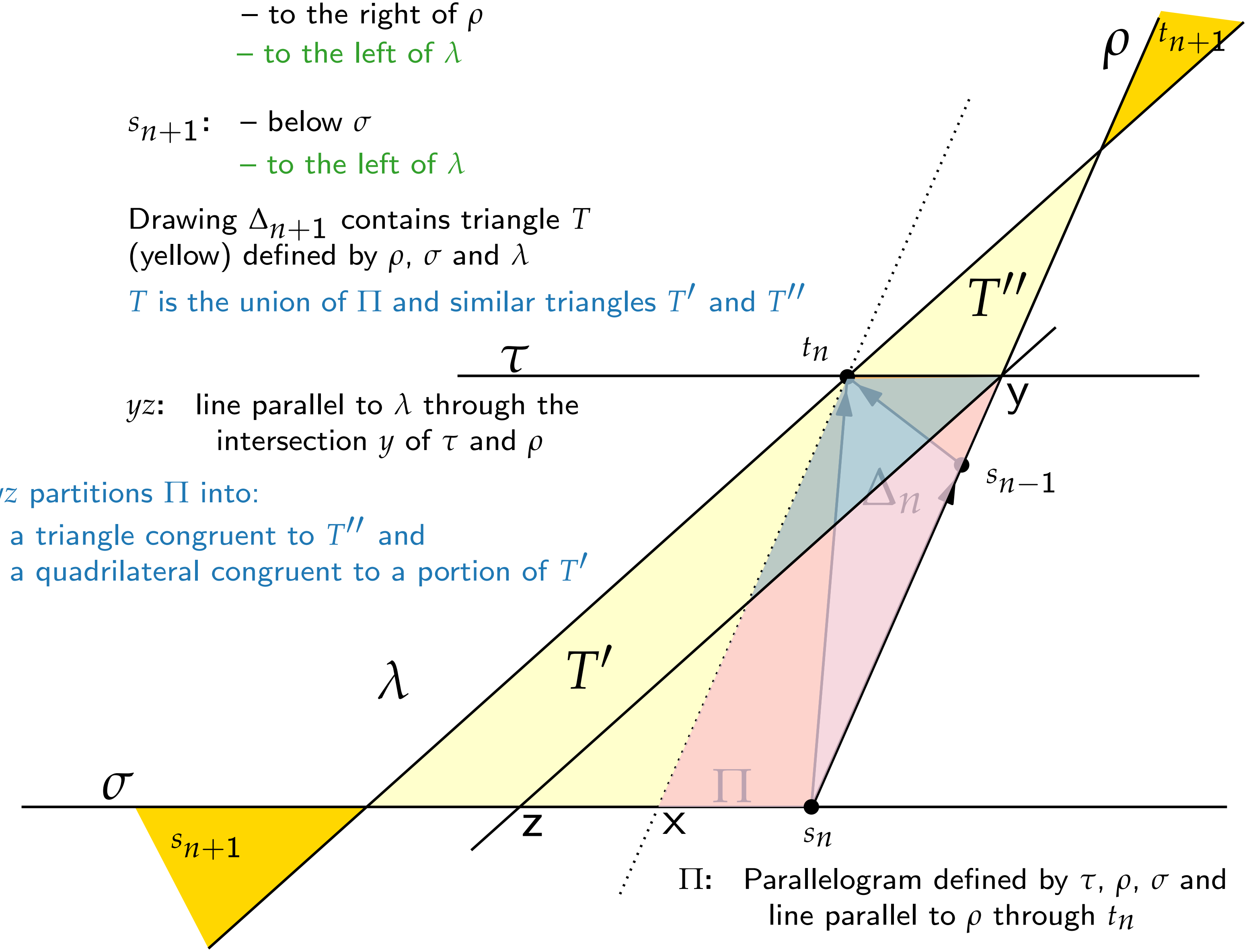

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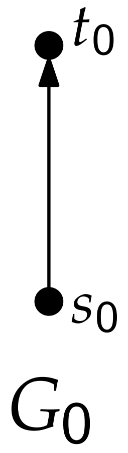

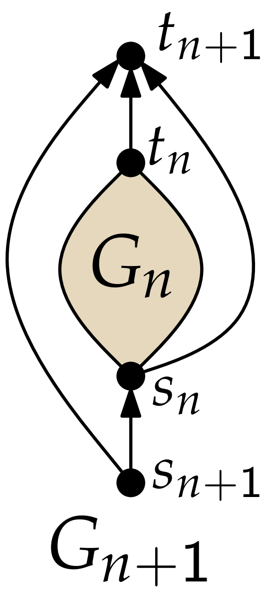

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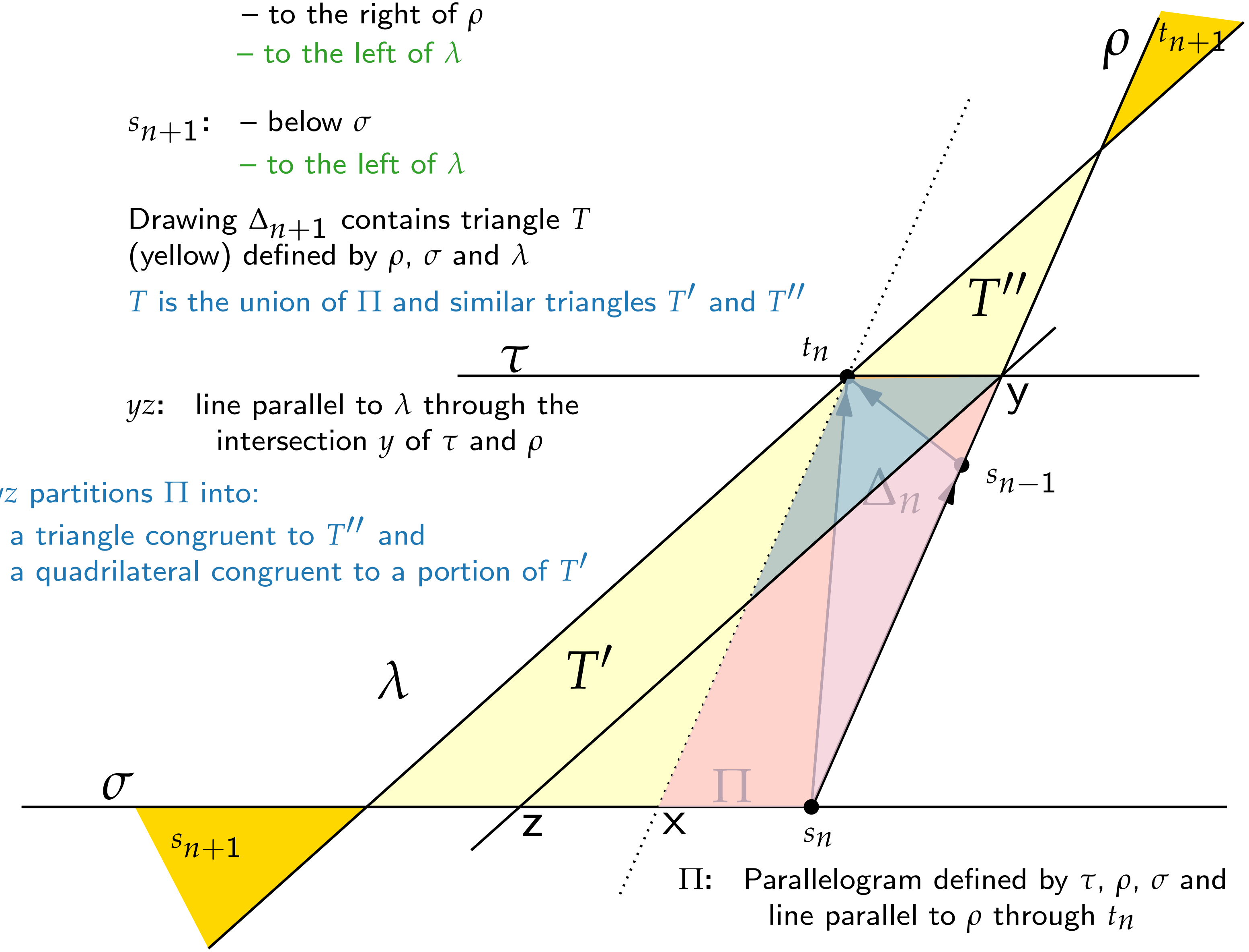

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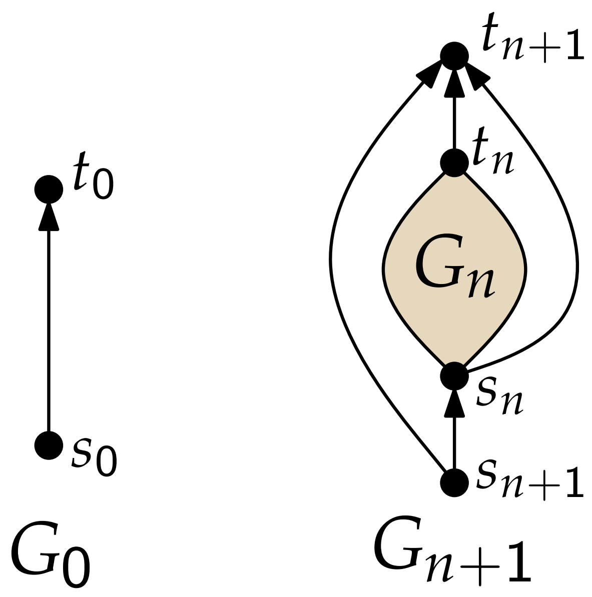

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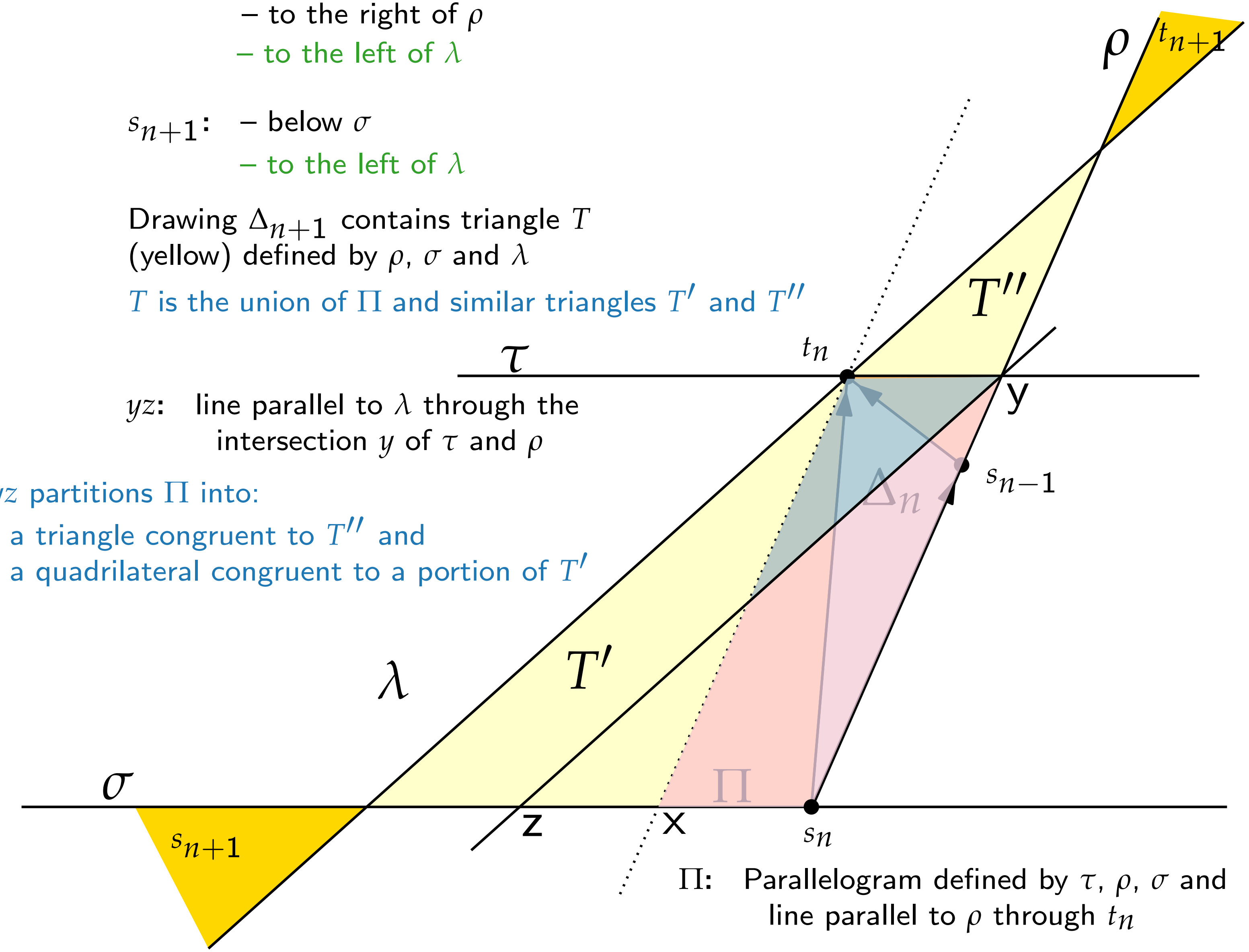

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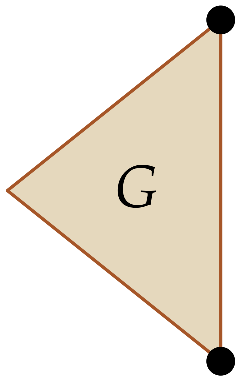
Series-parallel graphs – straight-line drawings

Divide & conquer algorithm using the decomposition tree

- Draw G inside a right-angled isosceles bounding triangle $\Delta(G)$ with no vertex placed at its right corner



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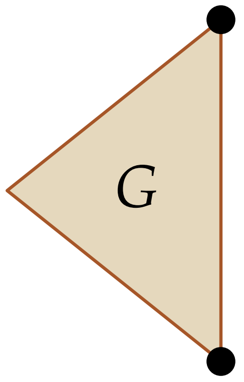

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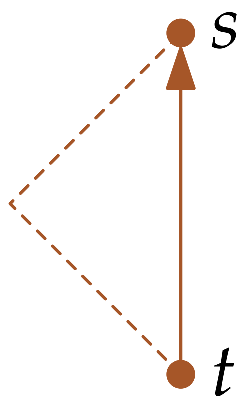

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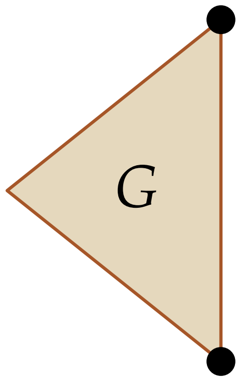

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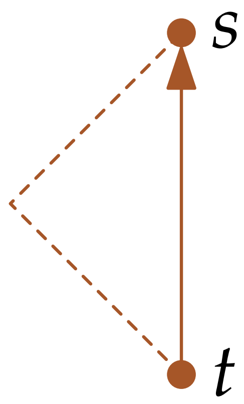

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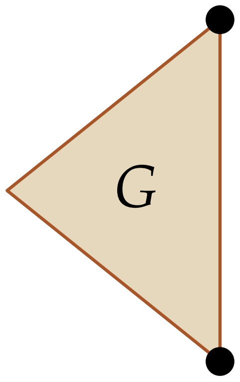

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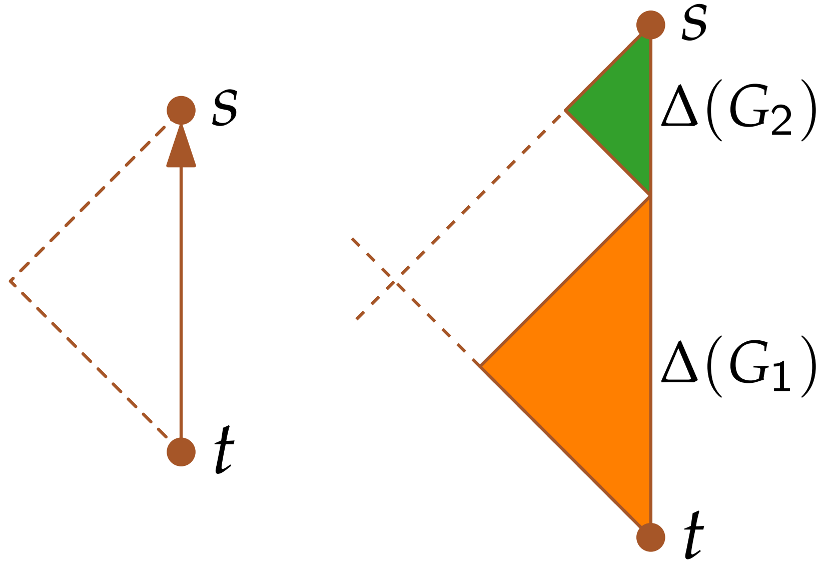

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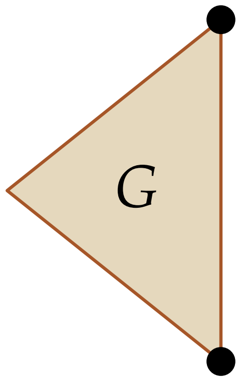

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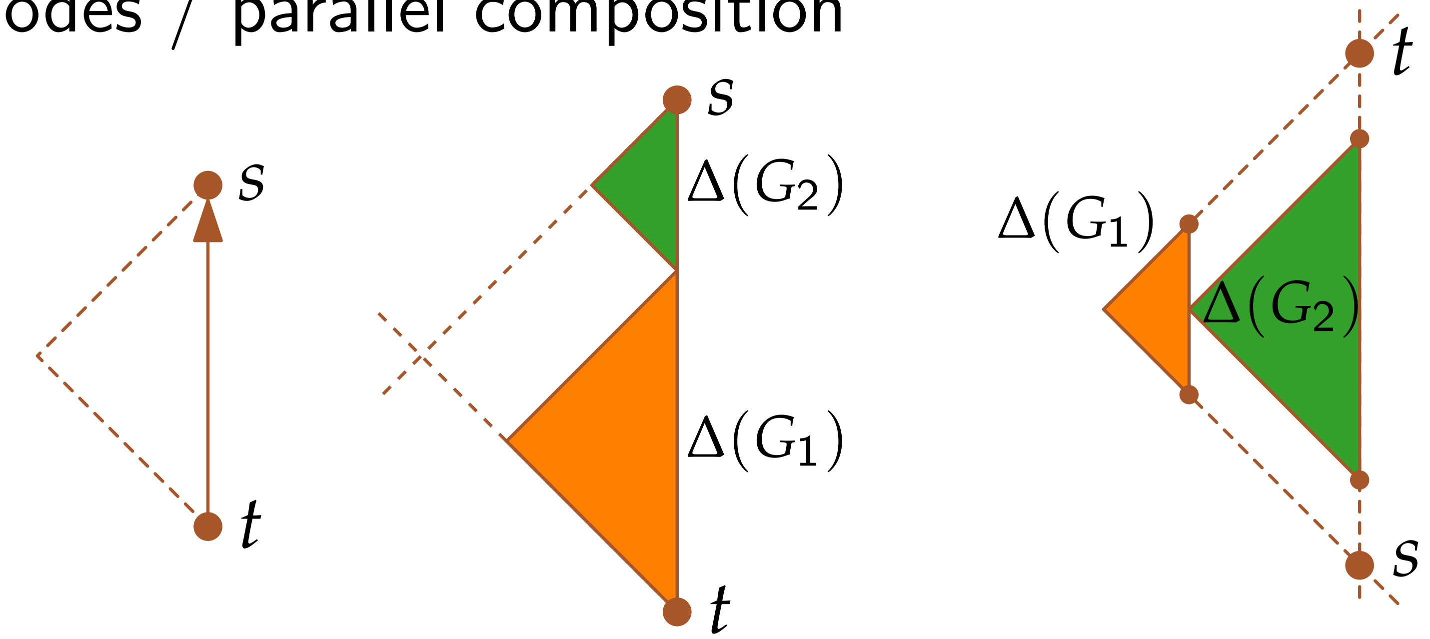

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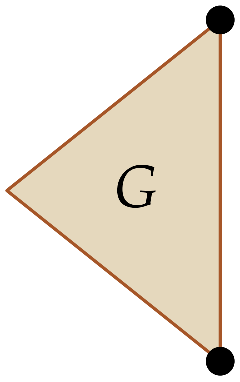

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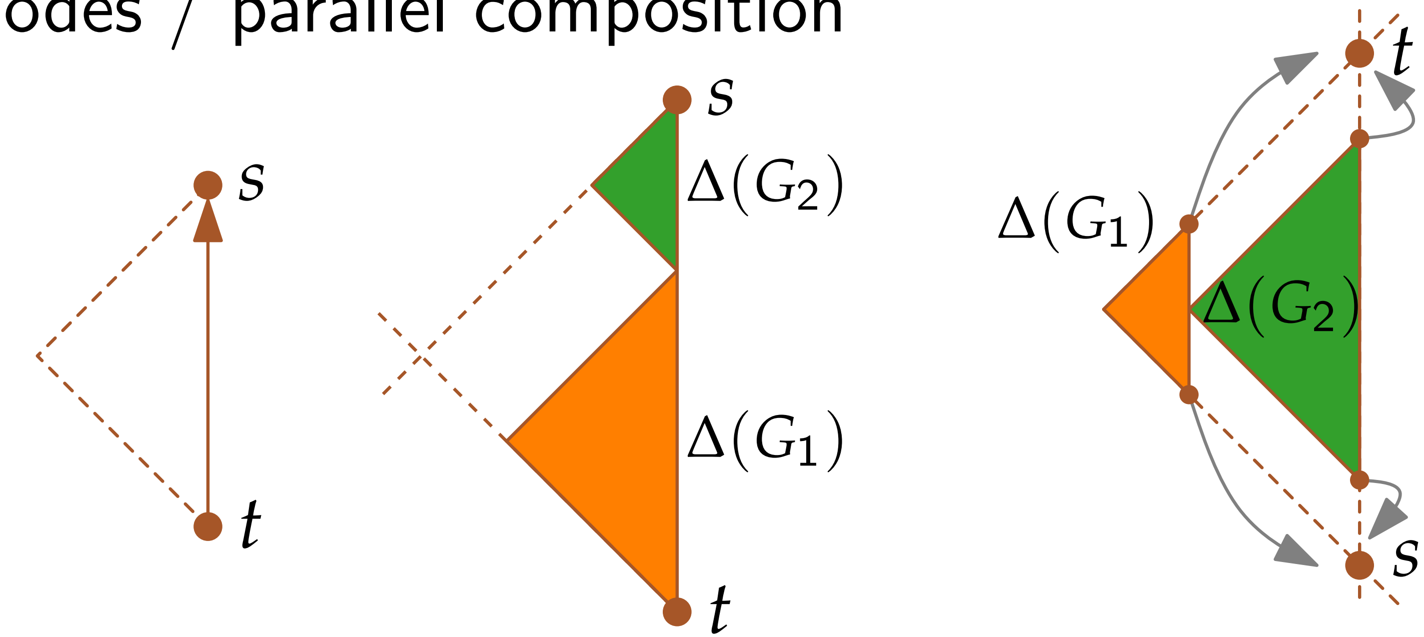

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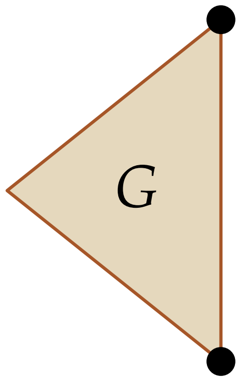

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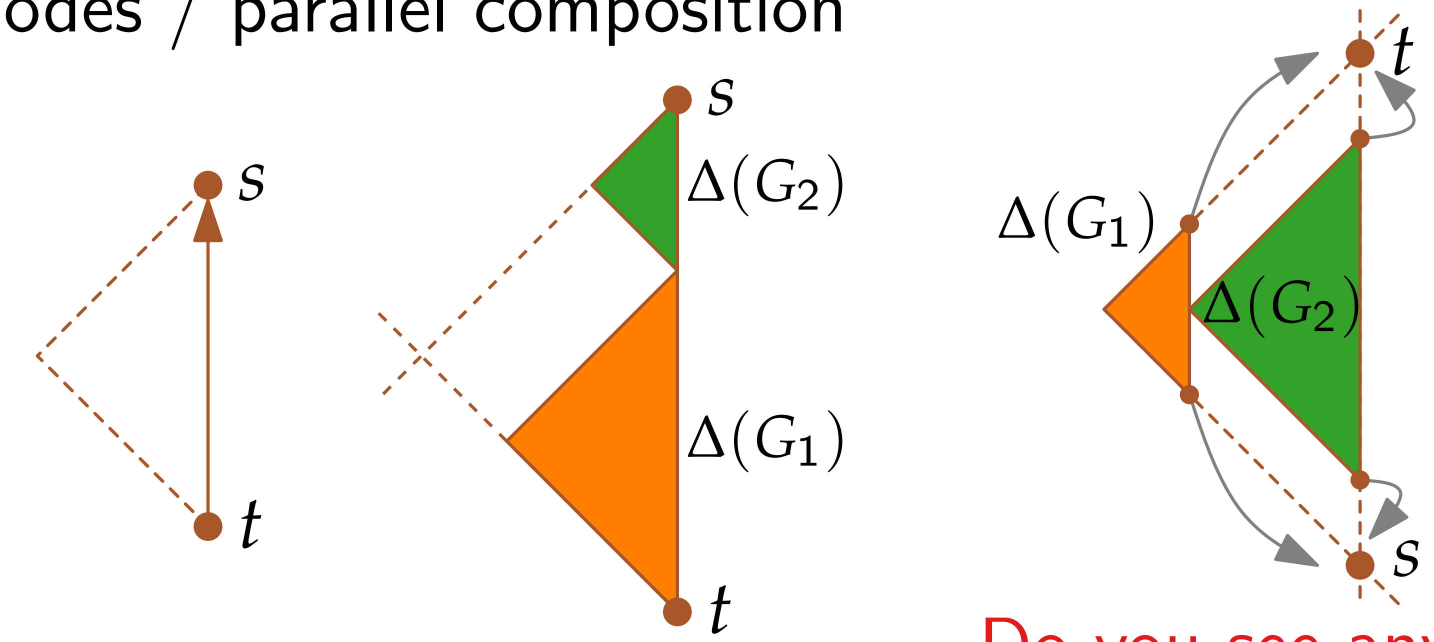
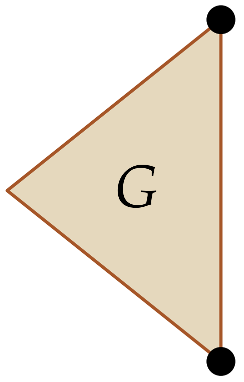


Do you see any problem?

Series-parallel graphs – straight-line drawings



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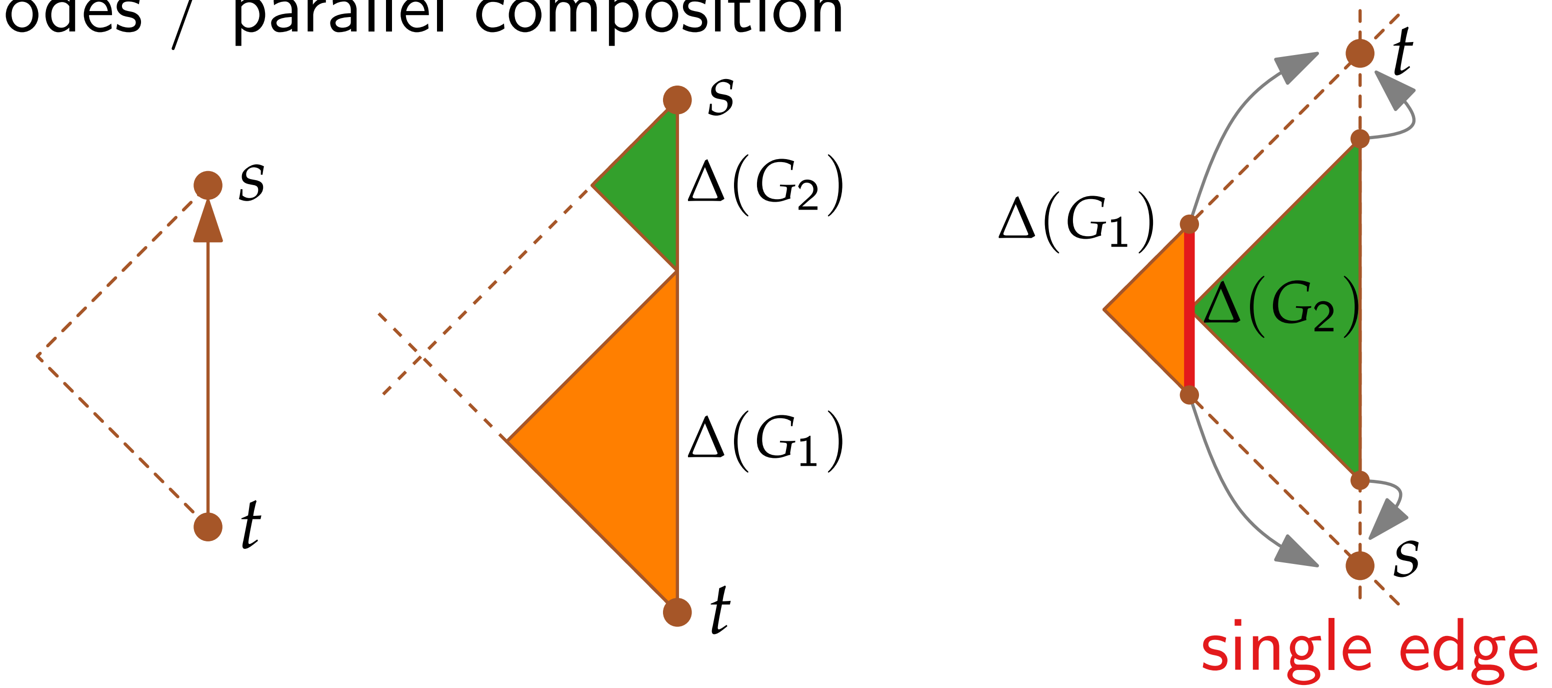

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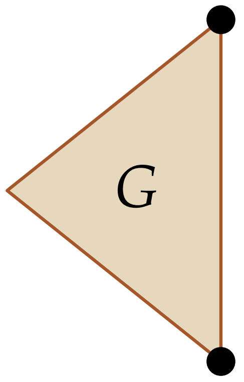

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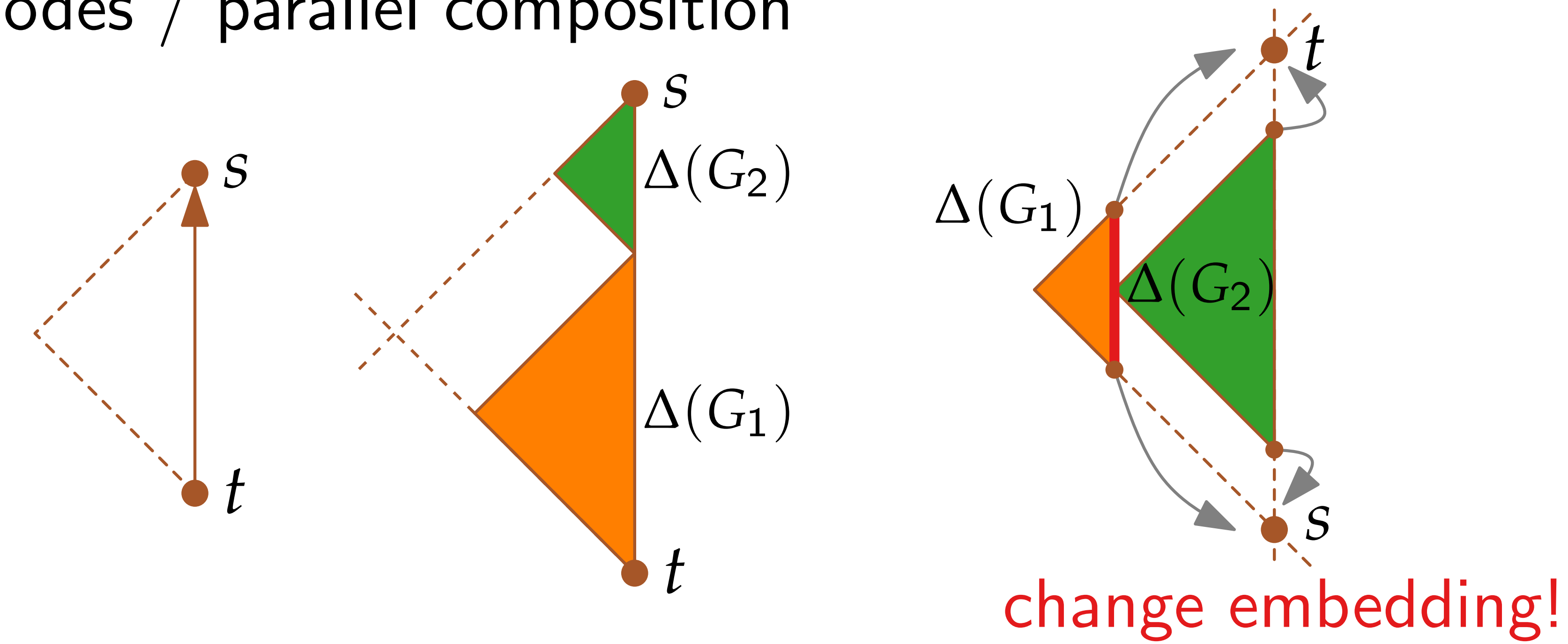

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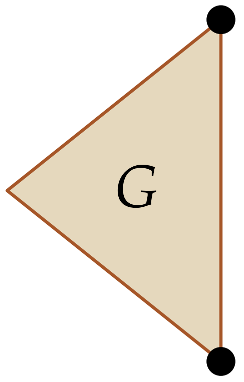

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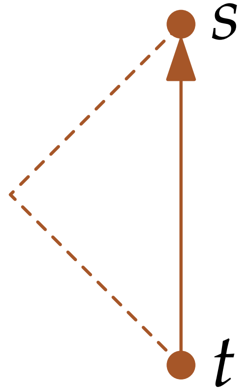
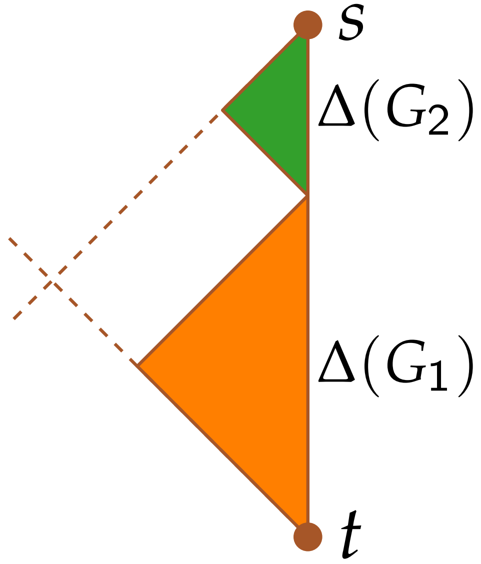
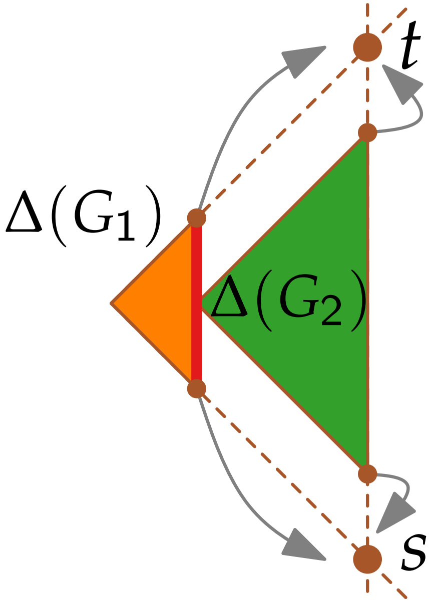
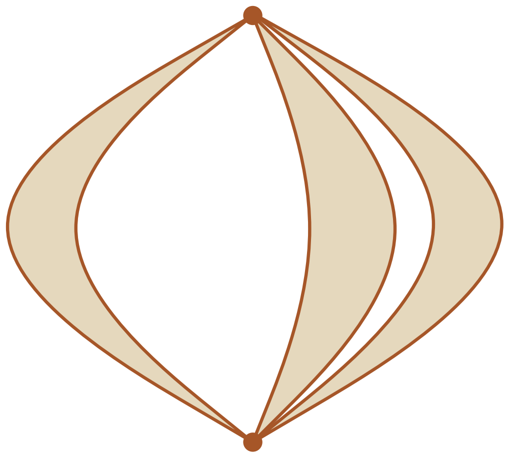






change embedding!



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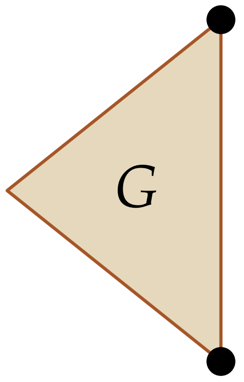

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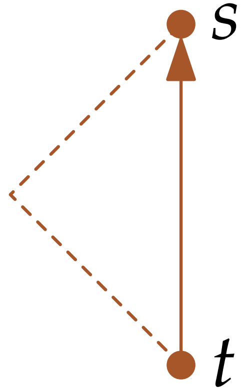

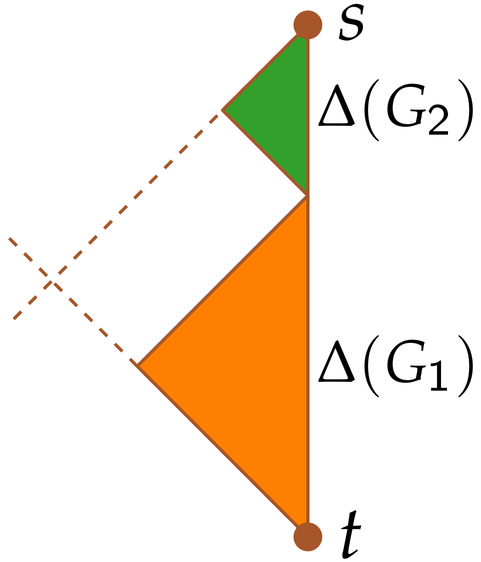

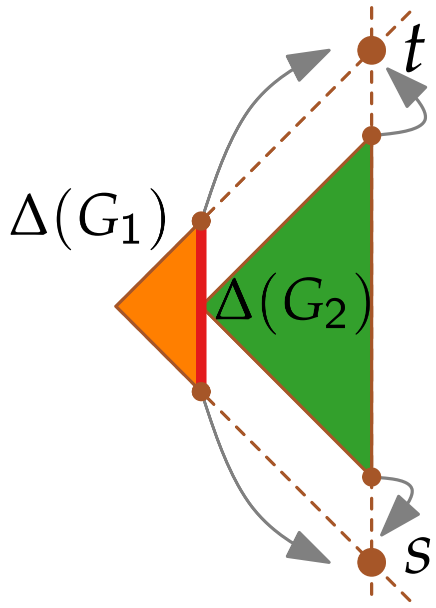

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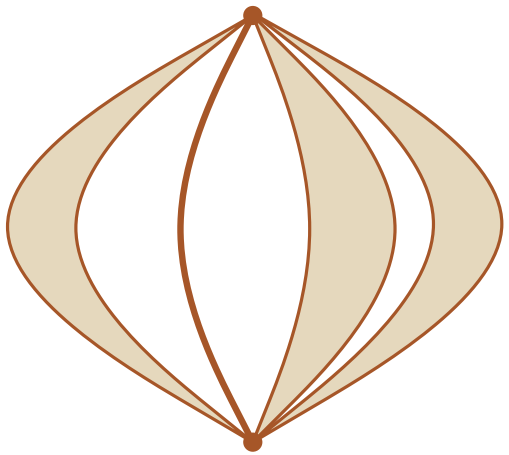

Series-parallel graphs – straight-line drawings

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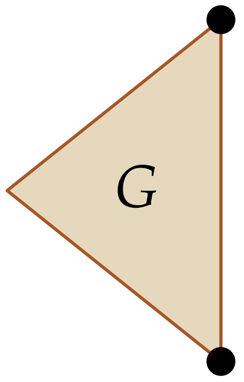

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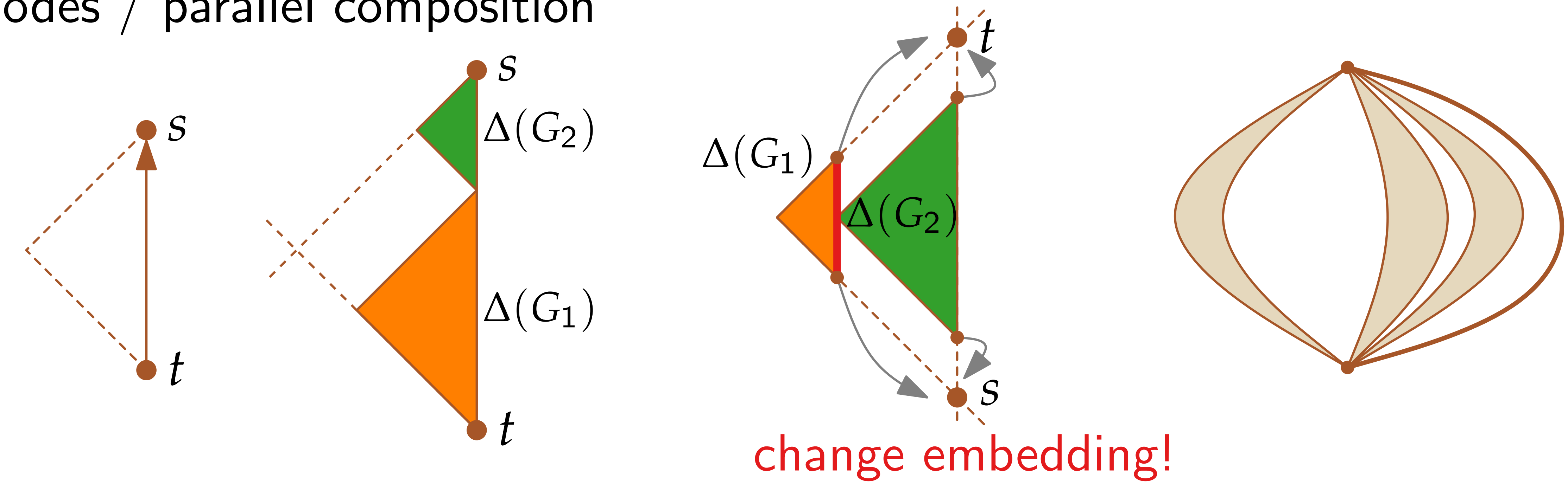

Series-parallel graphs – straight-line drawings

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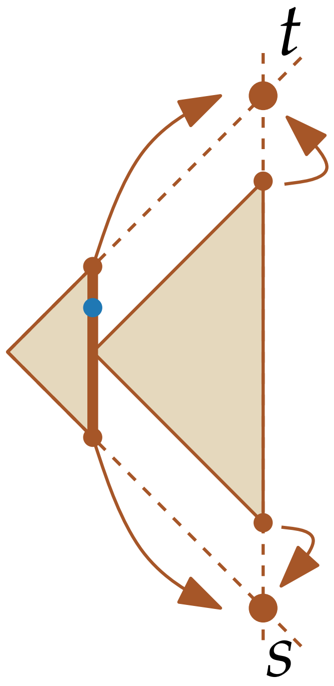

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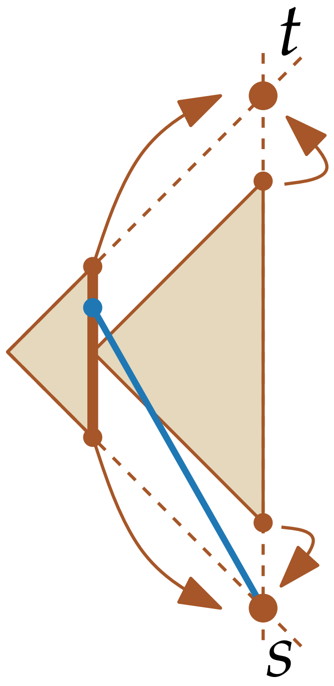

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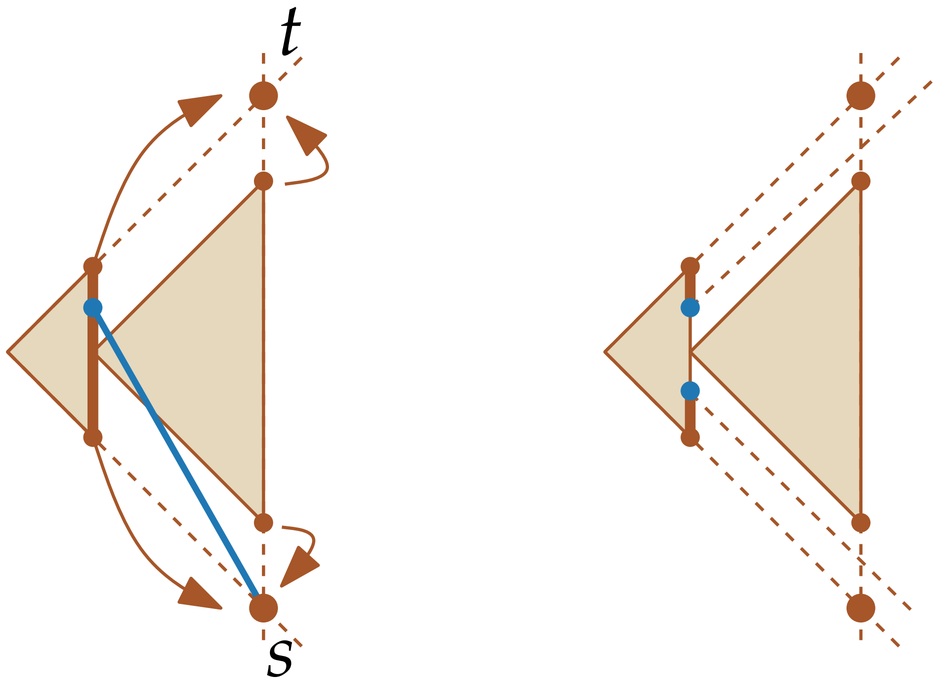

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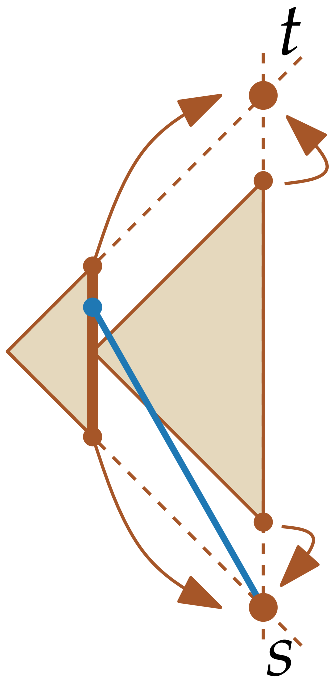

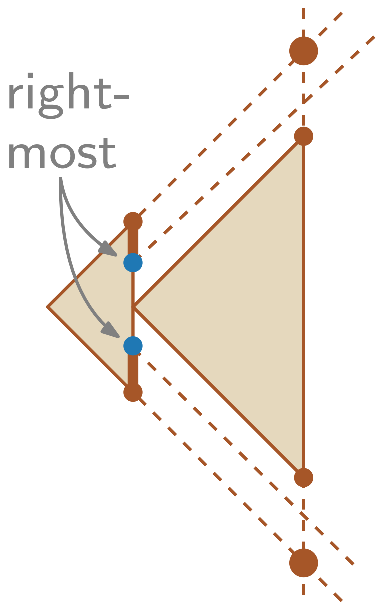

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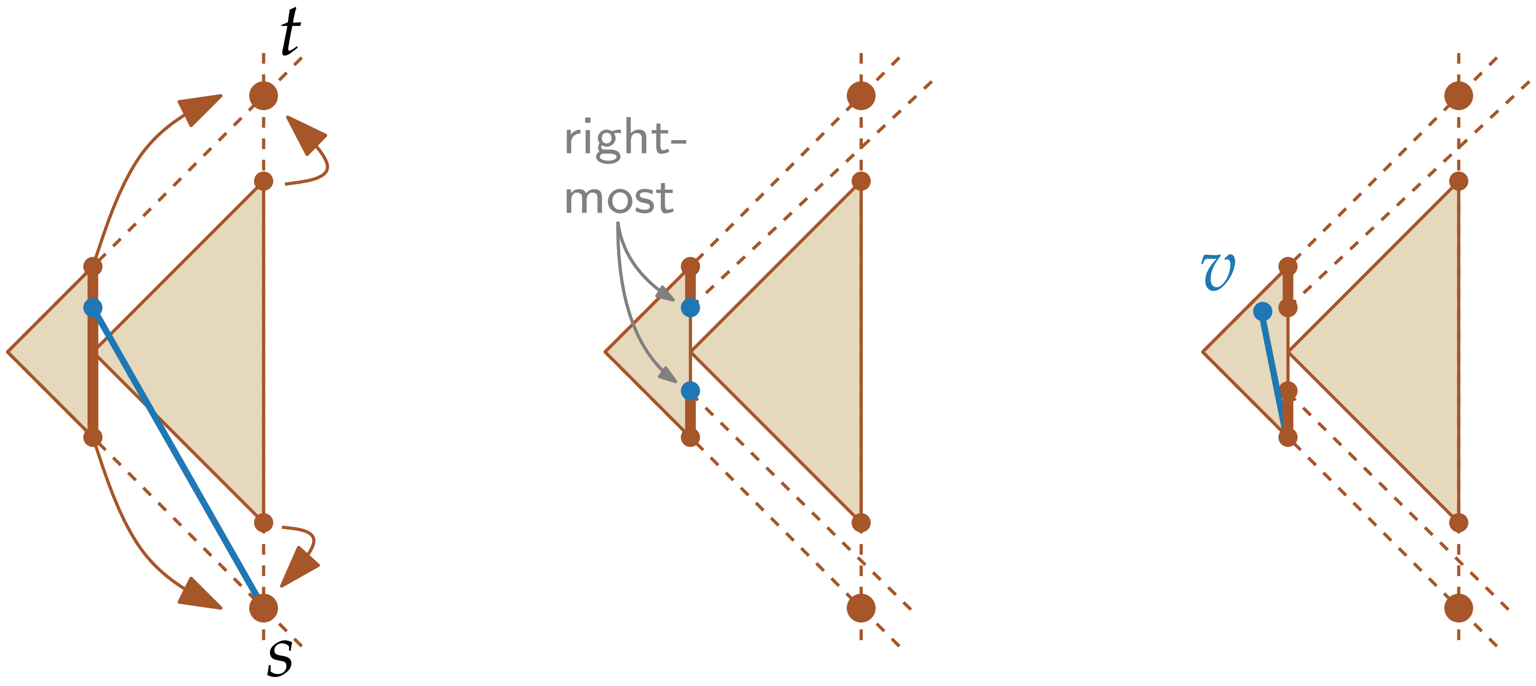

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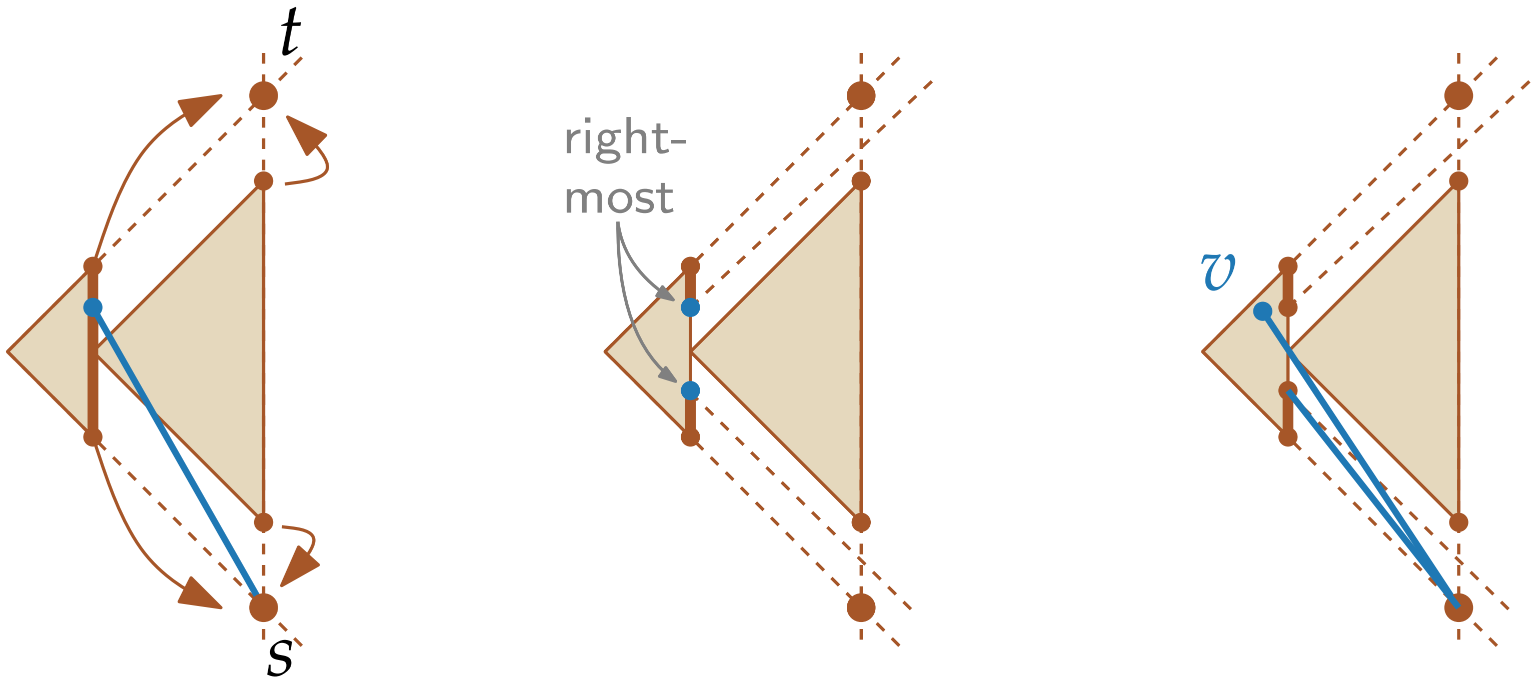

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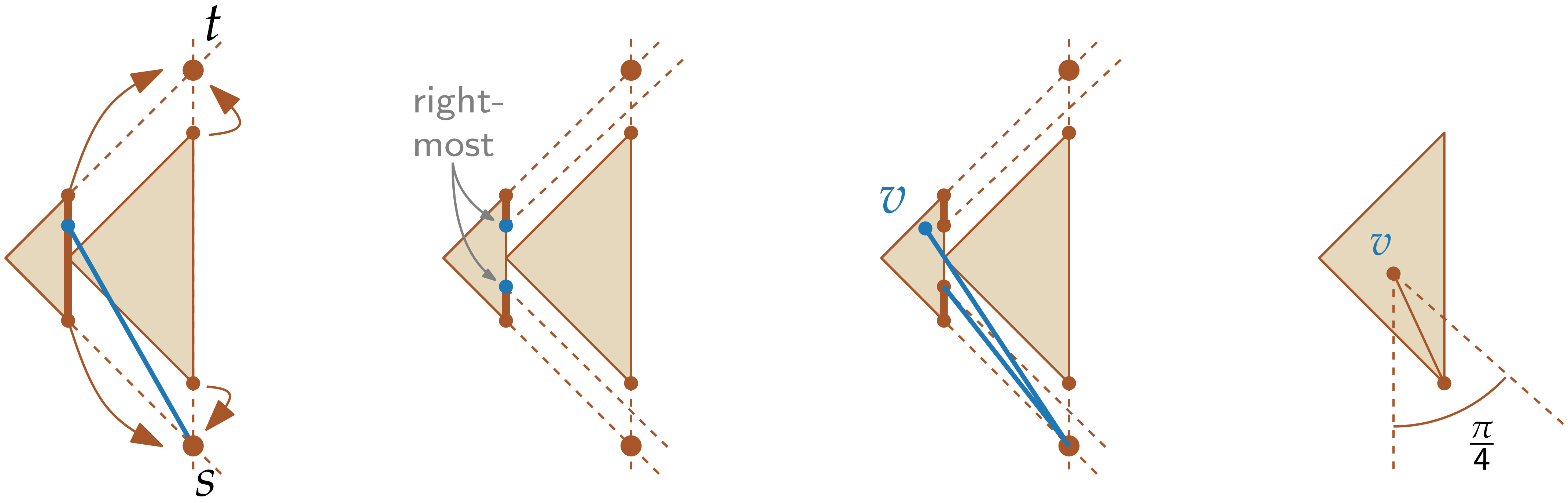

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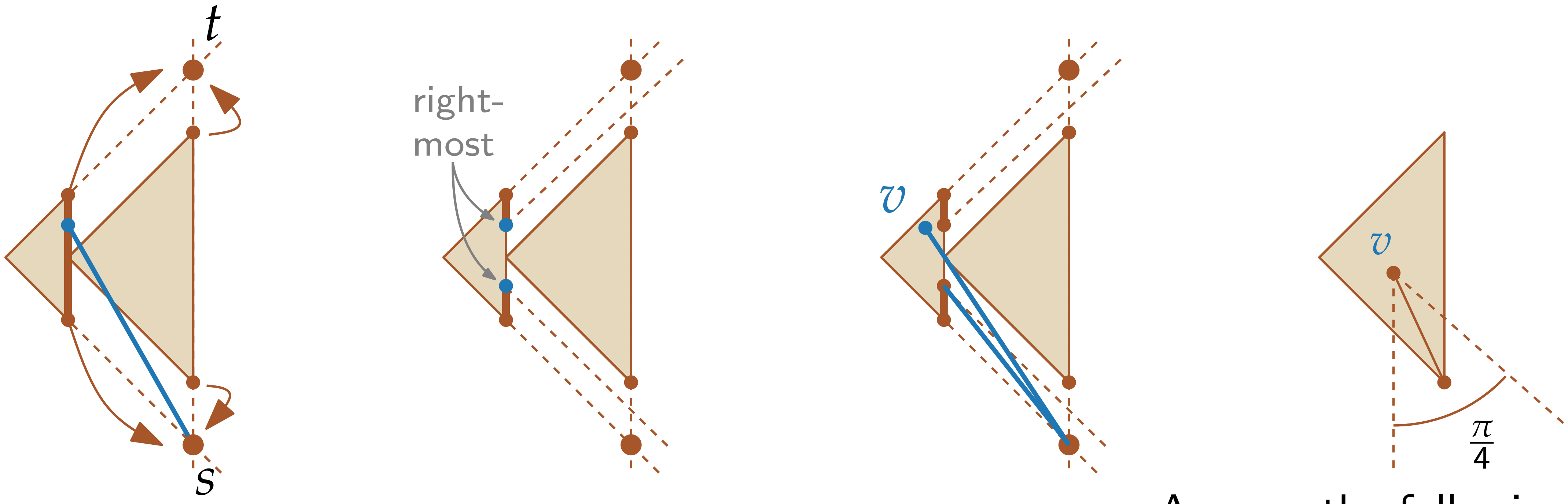

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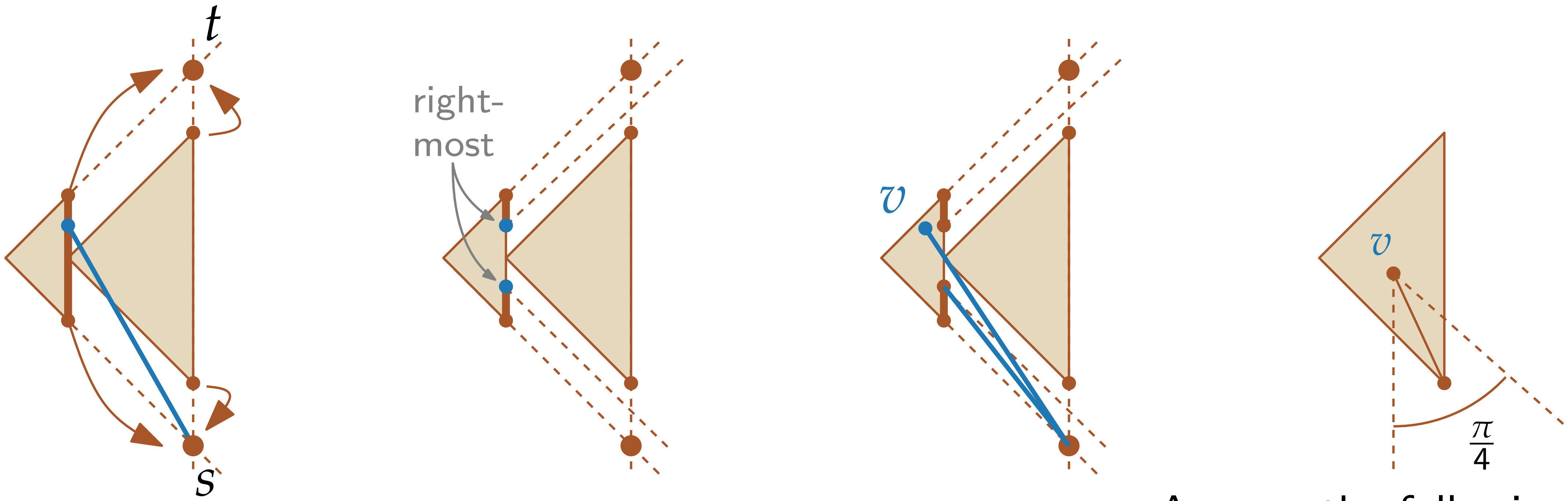

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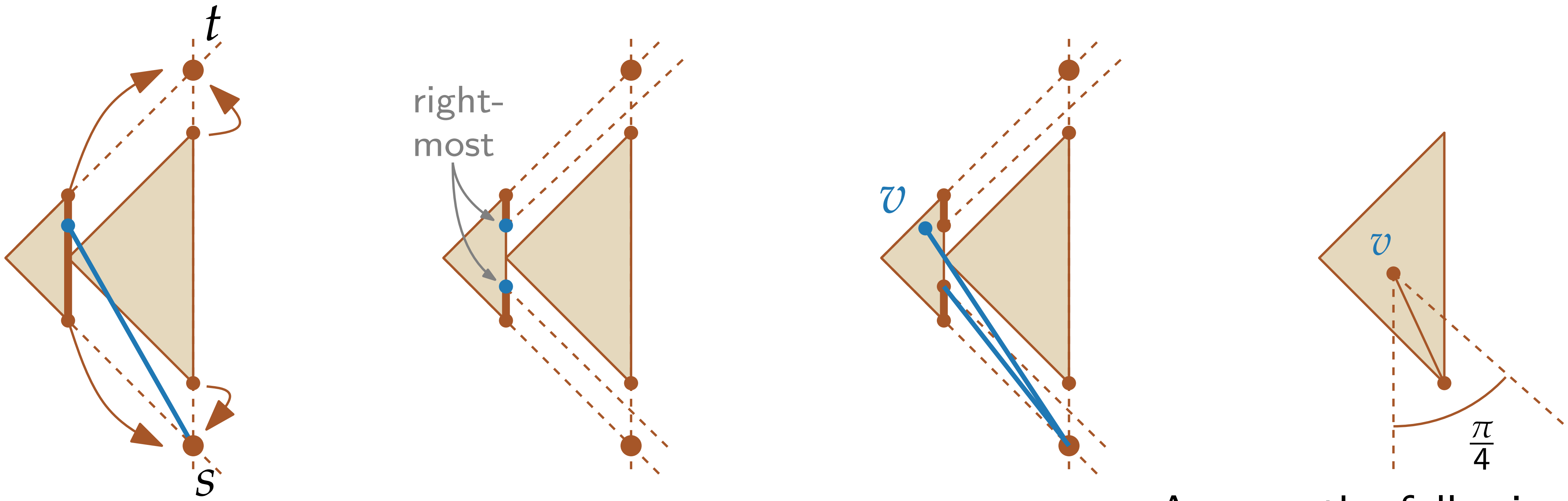

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Lemma.
The drawing produced by the algorithm is planar.

Series-parallel graphs – result

Theorem.
Let G be a series-parallel graph. Then G (with **variable embedding**) admits a drawing Γ that

- is upward planar and
- a straight-line drawing
- with area in $\mathcal{O}(n^2)$
  [$m \times 2m$, where m is the number of edges of G]
- Isomorphic components of G have congruent drawings up to translation.

Γ can be computed in $\mathcal{O}(n)$ time.

Literature

- [GD Ch. 3.2] for divide an conquer mehtods for series-parallel graphs.
- [BC+94] Bertolazzi, Cohen, Di Battista, Tamassia and Tollis, "How to draw a series-parallel digraph", Int. J. of Computational Geometry and Applications, Vol. 4, pp. 385-402, 1994.