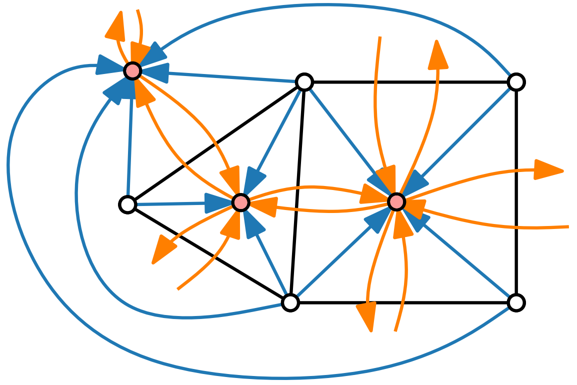
Visualisation of graphs

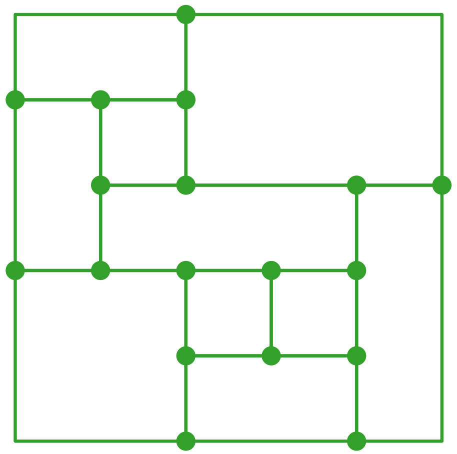
Orthogonal layouts

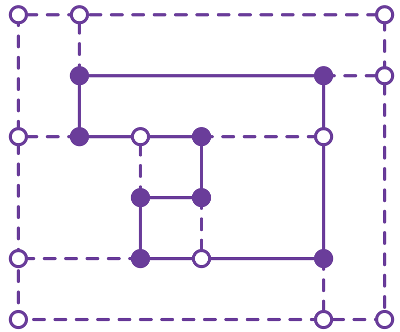
Flow methods

Antonios Symvonis · Chrysanthi Raftopoulou

Fall semester 2022



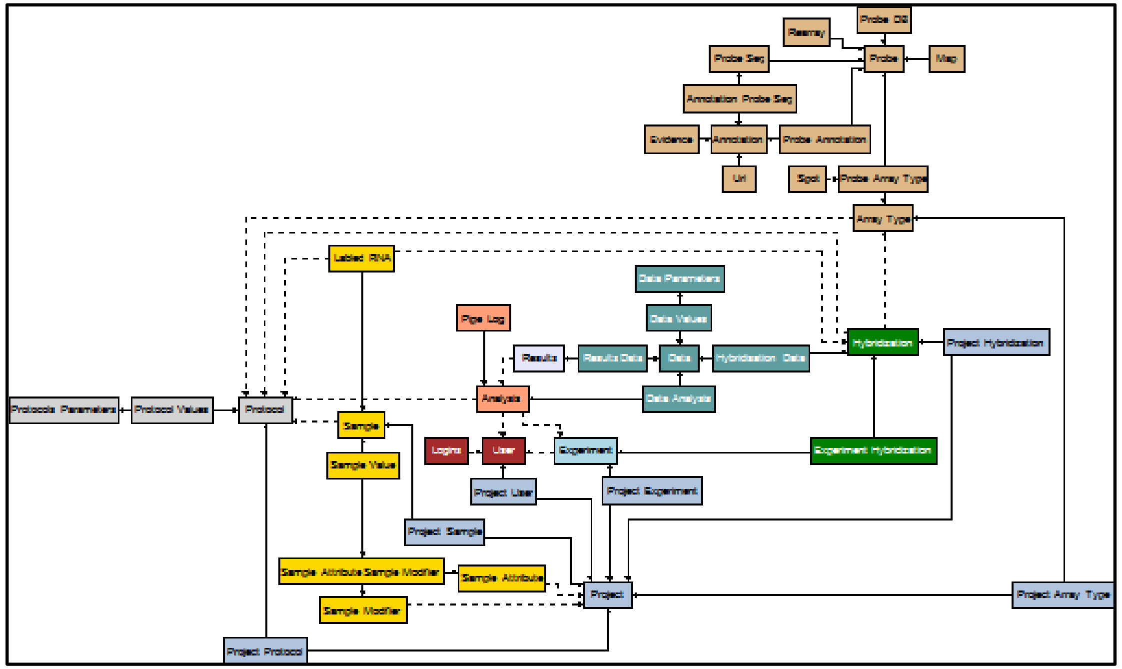




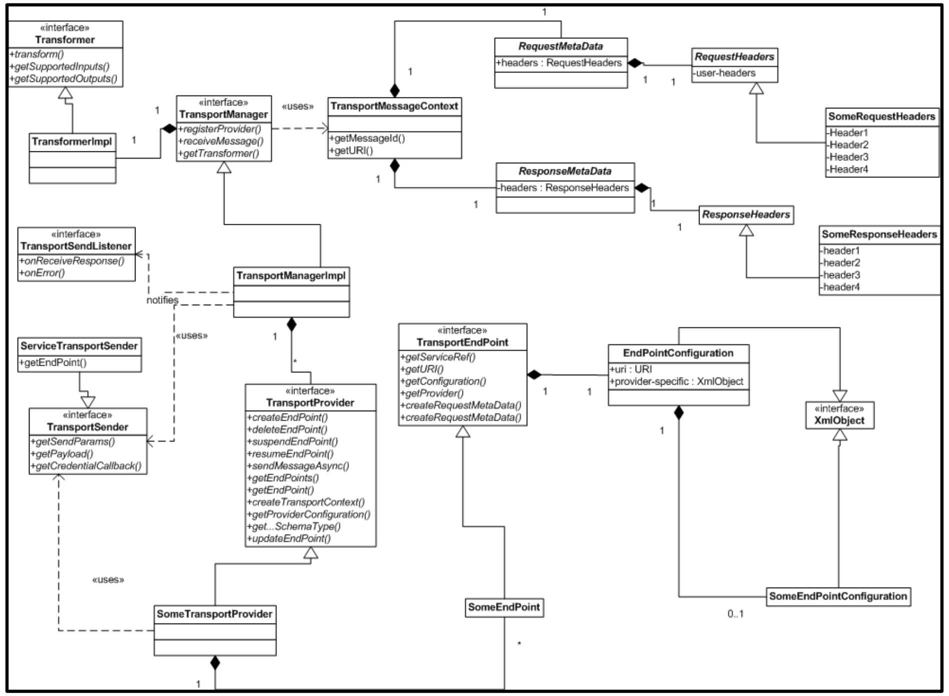
The original slides of this presentation were created by researchers at Karlsruhe Institute of Technology (KIT), TU Wien, U Wuerzburg, U Konstanz, ...
The original presentation was modified/updated by A. Symvonis and C. Raftopoulou

Orthogonal layout – applications

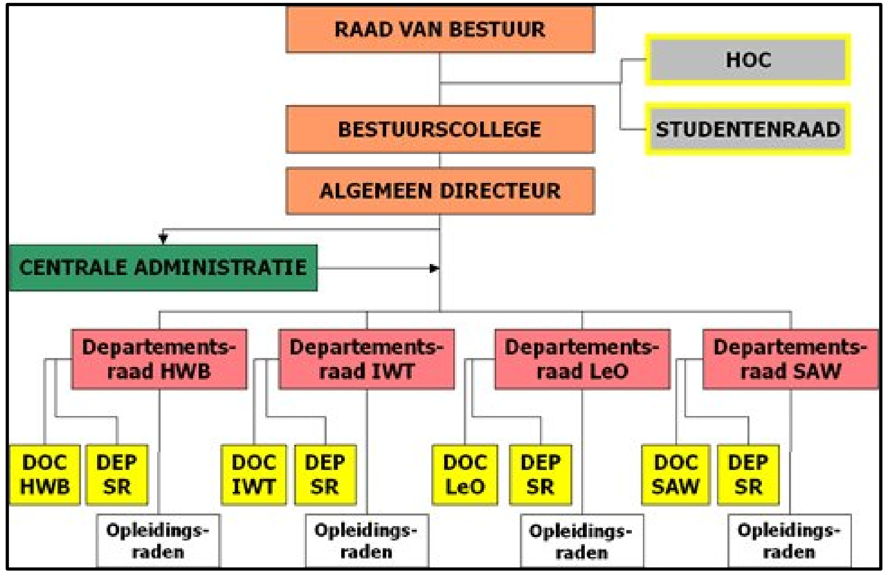
ER diagram in OGDF



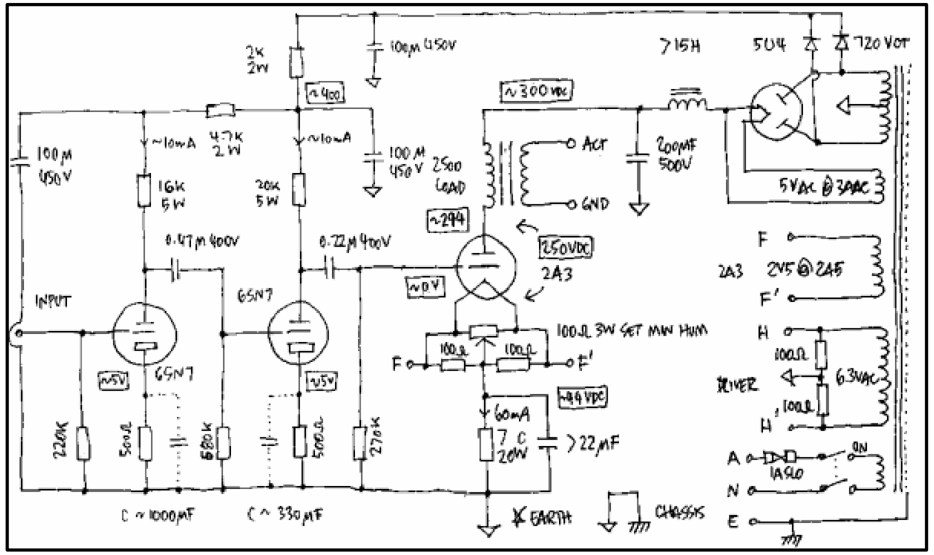
UML diagram by Oracle



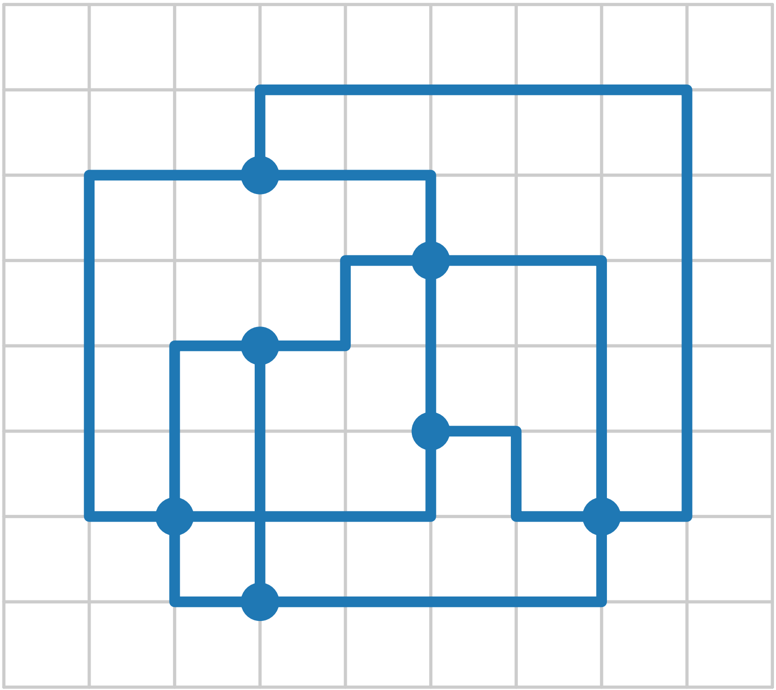
Organigram of HS Limburg



Circuit diagram by Jeff Atwood



Orthogonal layout – definition



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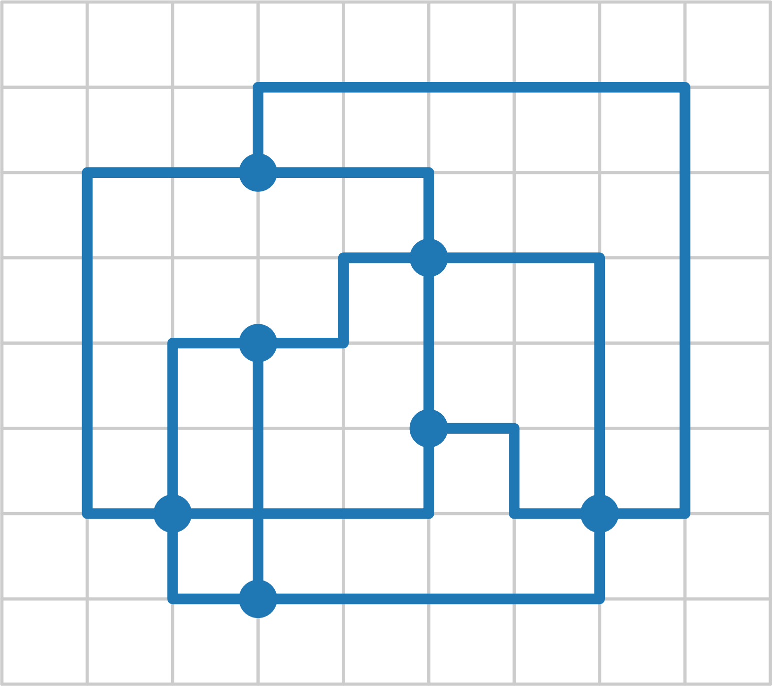

Definition.
A drawing Γ of a graph $G = (V, E)$ is called **orthogonal** if

- veritices are drawn as points on a grid,
- each edge is represented as a sequence of alternating horizontal and vertical segments, and
- pairs of edges are disjoint or cross orthogonally.

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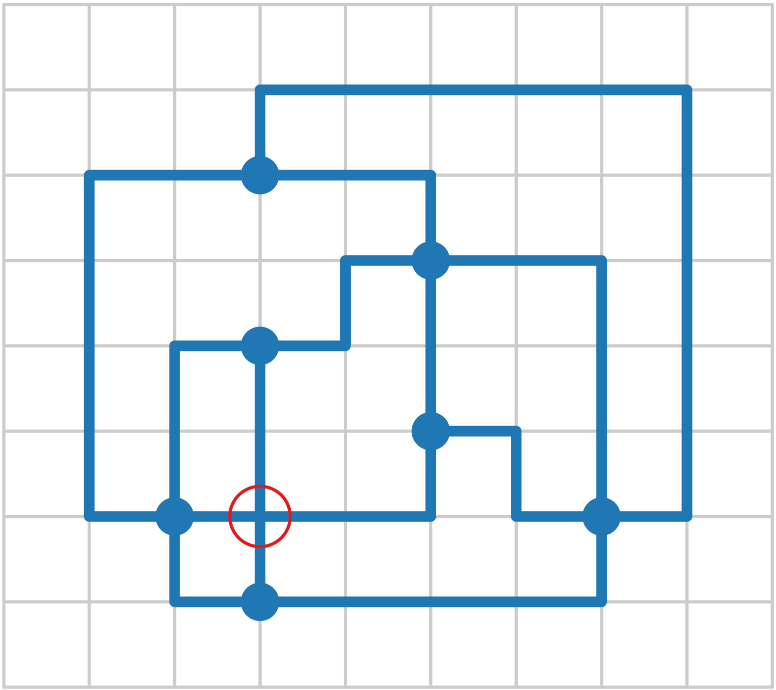

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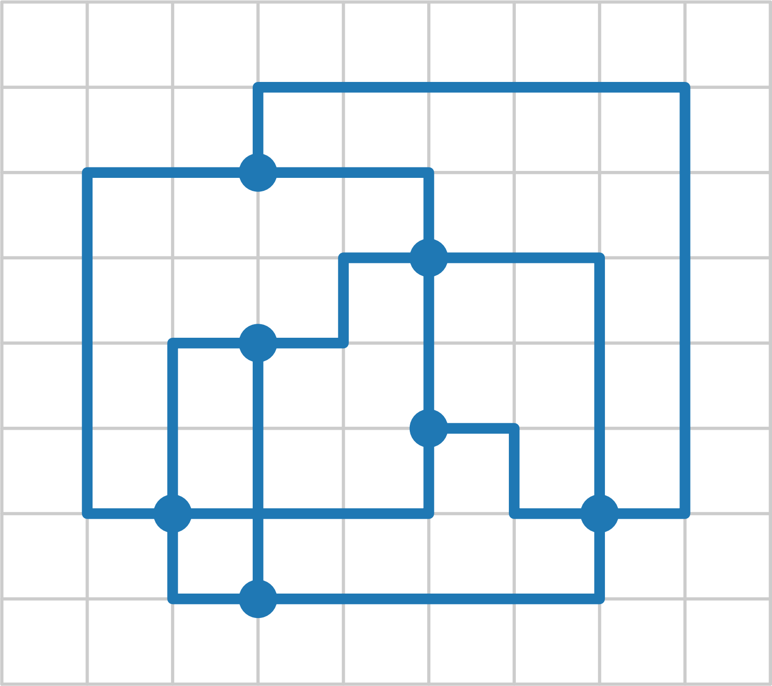

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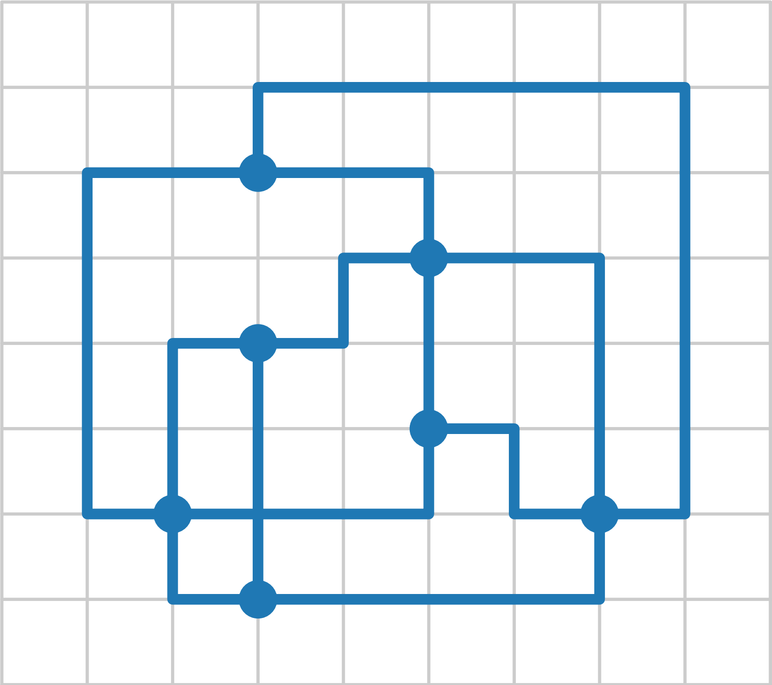
Observations.

- Edges lie on grid $\Rightarrow$ **bends** lie on grid points
- Max degree of each vertex is at most 4

Orthogonal layout – definition



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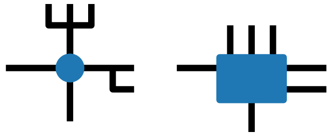
  - Otherwise 

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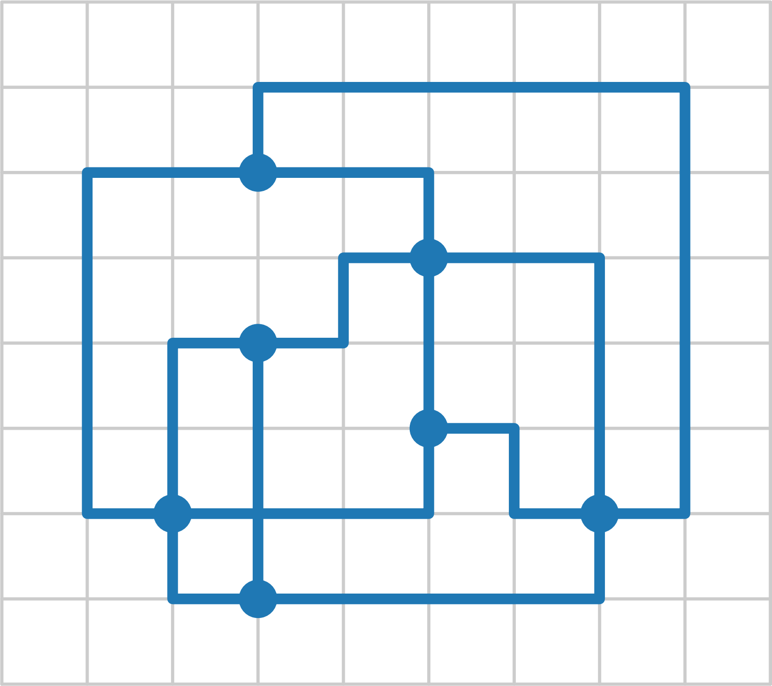

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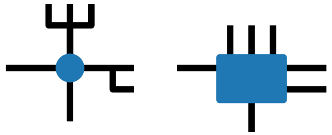
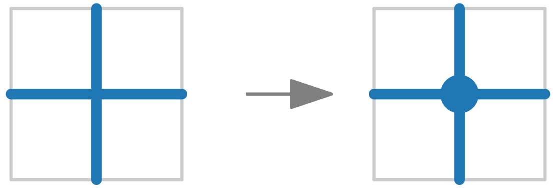
  - Otherwise 

Planarisation.

- Fix embedding
- Crossings become vertices



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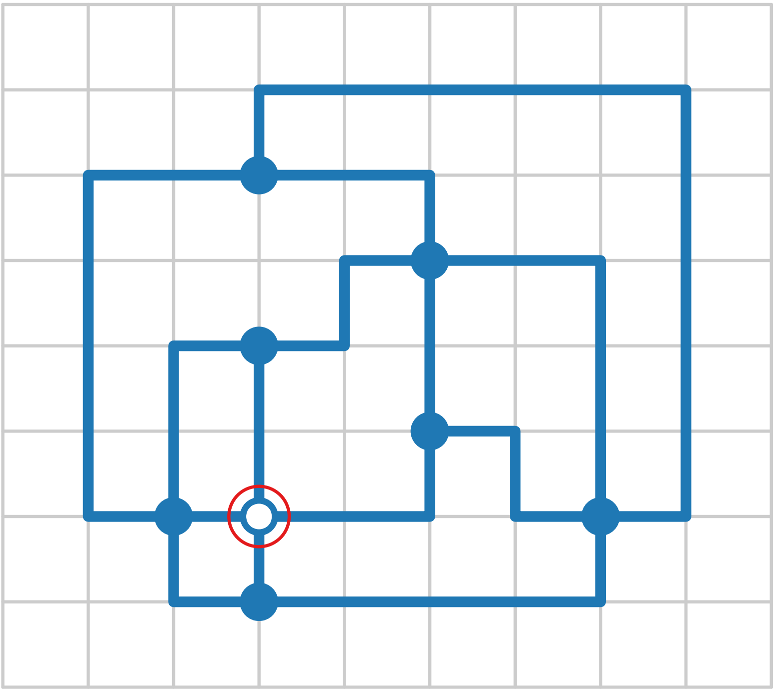

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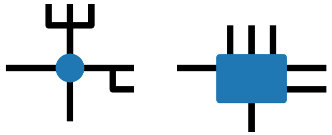
  - Otherwise 

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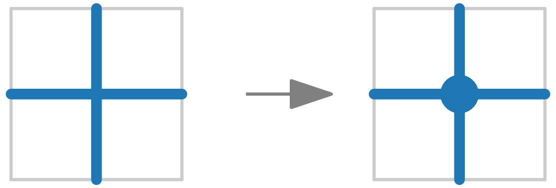

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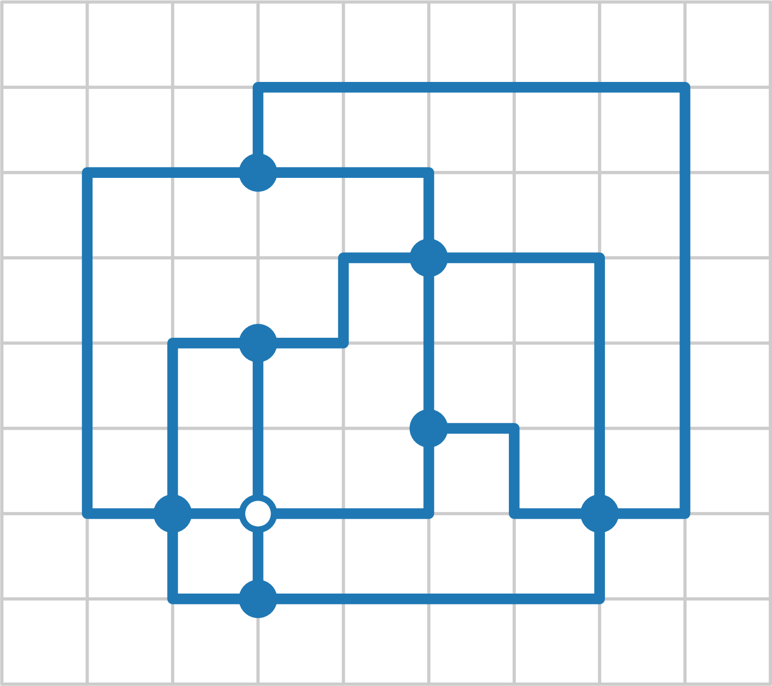

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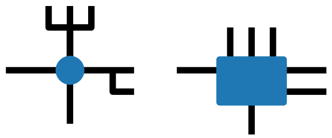
  - Otherwise 

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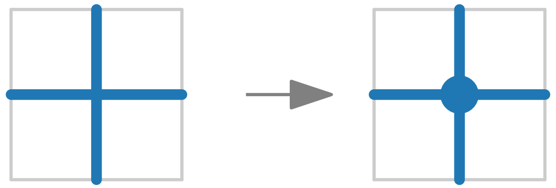

Aesthetic criteria.

- Number of bends
- Length of edges
- Width, height, area
- Monotonicity of edges
- ...

Topology - Shape - Metrics

Three-step approach:
[Tam87]

$V = \{v_1, v_2, v_3, v_4\}$
$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$

Topology - Shape - Metrics

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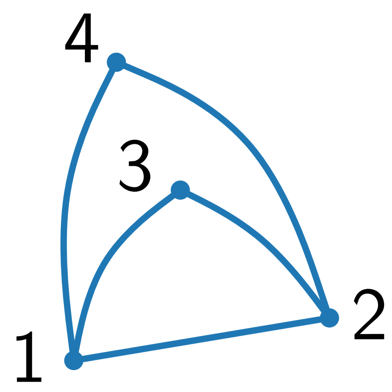
reduce crossings

combinatorial embedding/ planarisation



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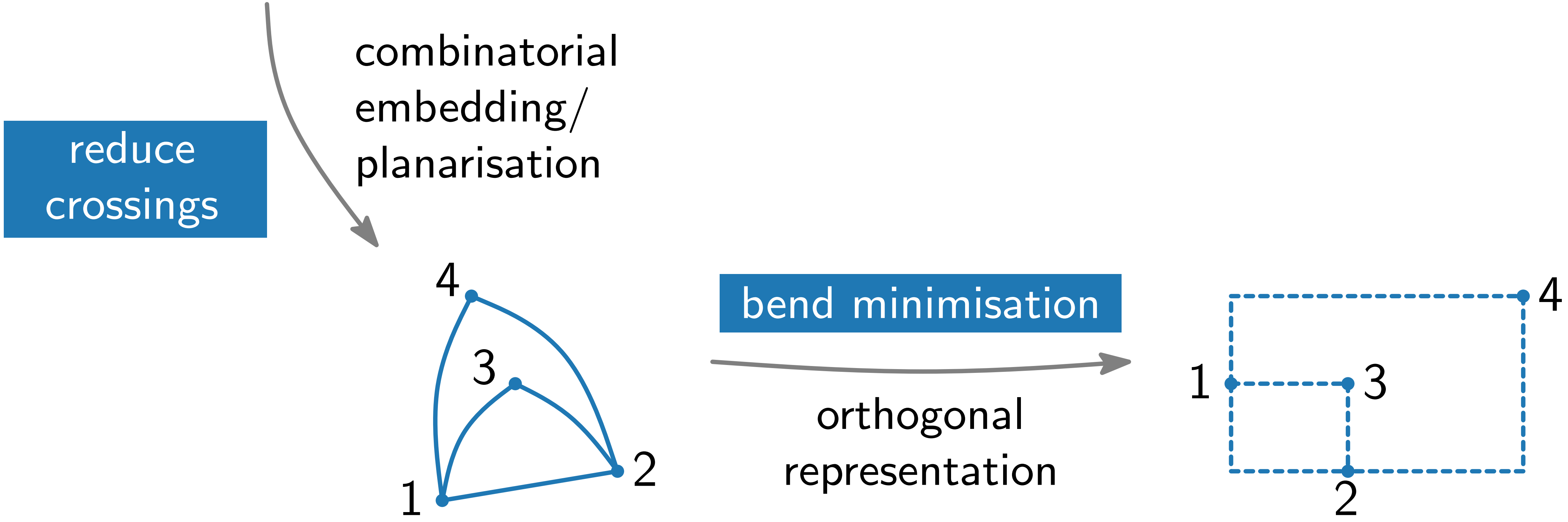

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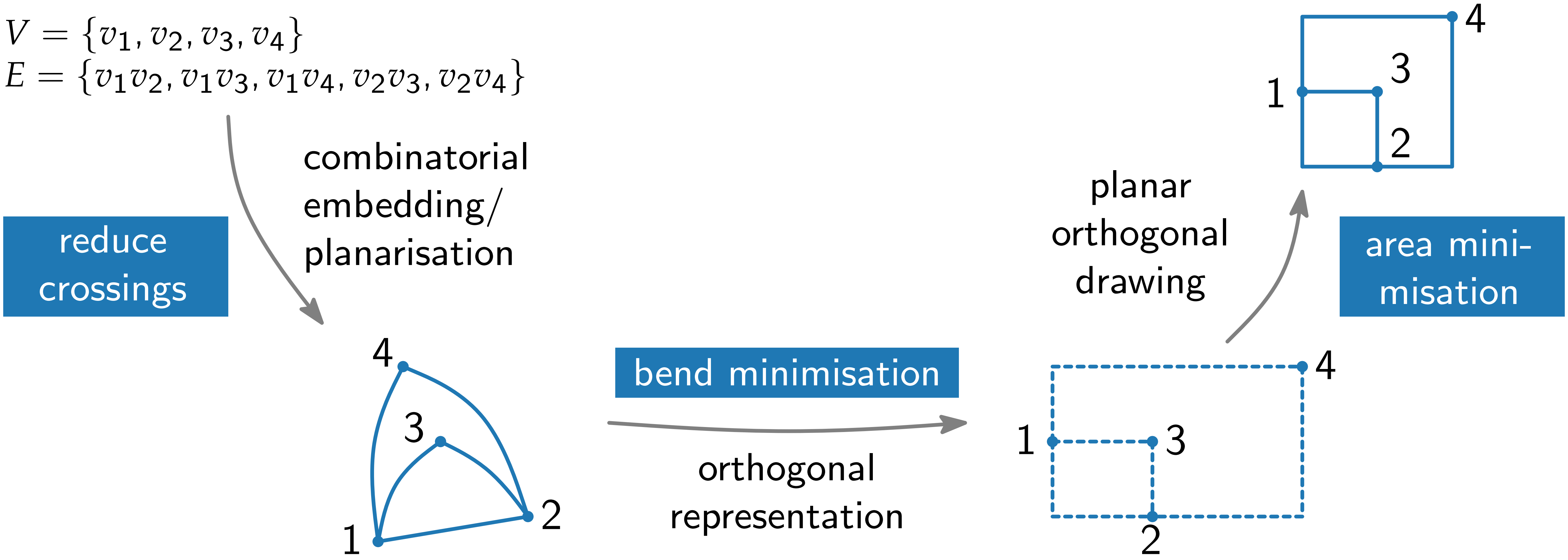

Orthogonal representation

Idea.

Describe orthogonal drawing combinatorially.

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Let $G = (V, E)$ be a plane graph with faces F and outer face f_0.

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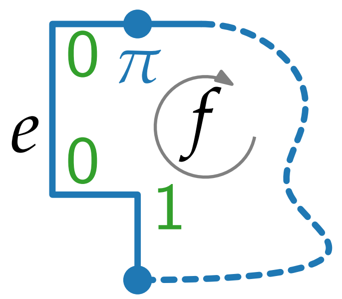
Definitions.

Let $G = (V, E)$ be a plane graph with faces F and outer face f_0.

- Let e be an edge with the face f to the right.
  An **edge description** of e wrt f is a triple (e, δ, α) where
  - δ is a sequence of $\{0,1\}^*$ ($0 =$ right bend, $1 =$ left bend)
  - α is angle $\in \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ between e and next edge e'



$(e, 100, \pi)$

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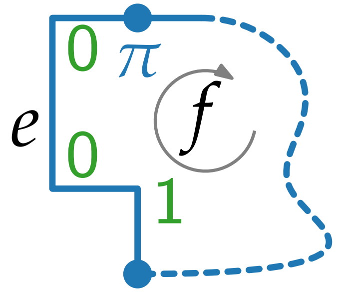
- A **face representation** $H(f)$ of f is a clockwise ordered sequence of edge descriptions (e, δ, α).



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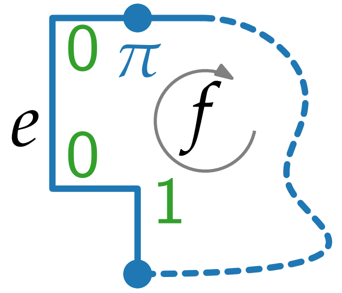

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- A **face representation** $H(f)$ of f is a clockwise ordered sequence of edge descriptions (e, δ, α).

- An **orthogonal representation** $H(G)$ of G is defined as

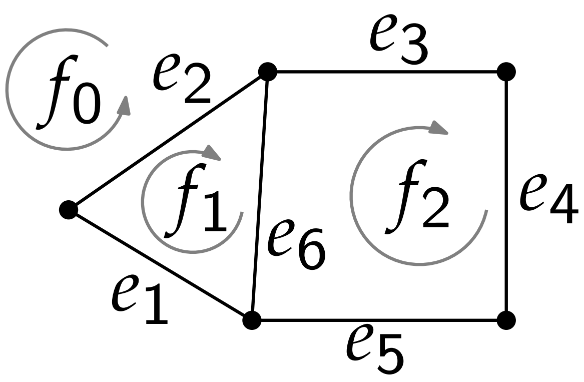
$$H(G) = \{H(f) \mid f \in F\}.$$

Orthogonal representation – example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$

$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$

$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$



Combinatorial "drawing" of $H(G)$?

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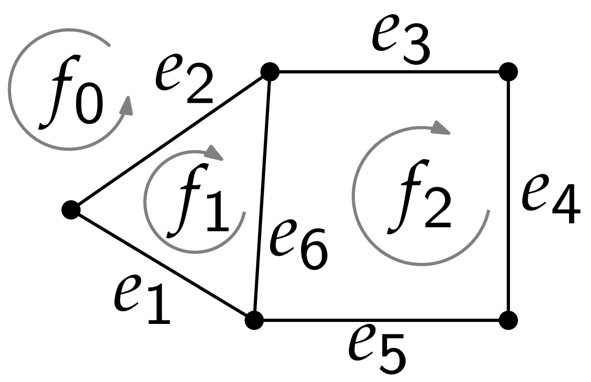

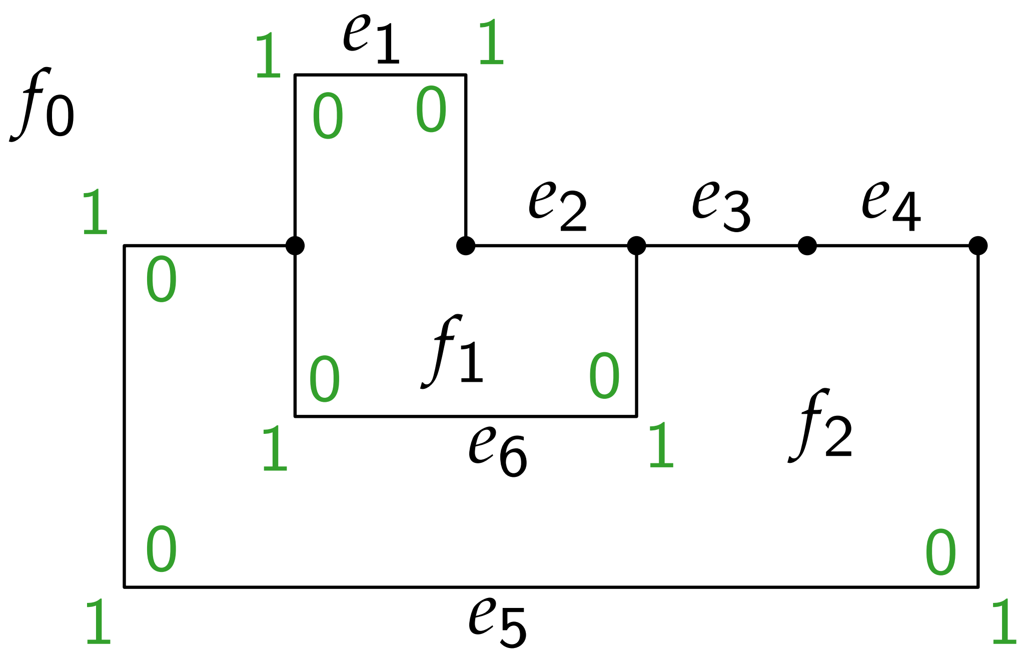

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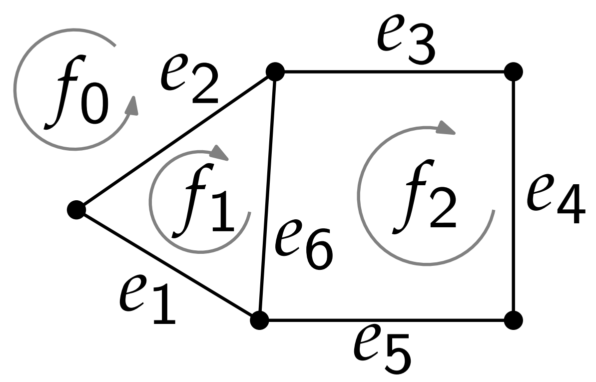

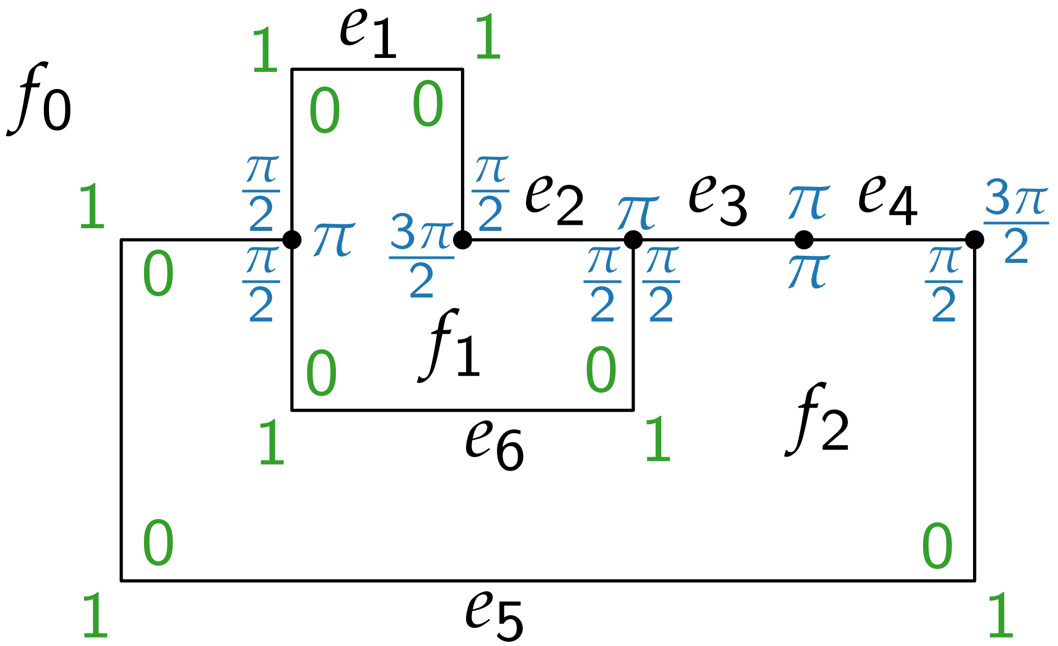

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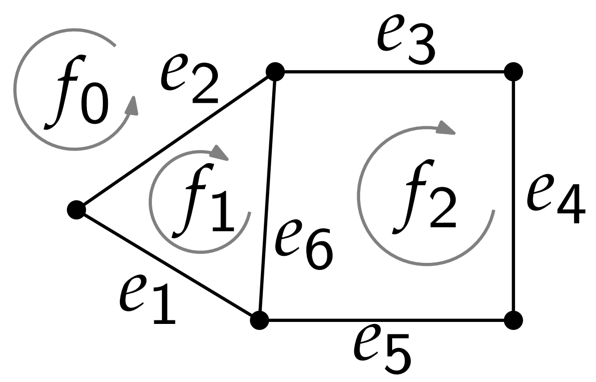

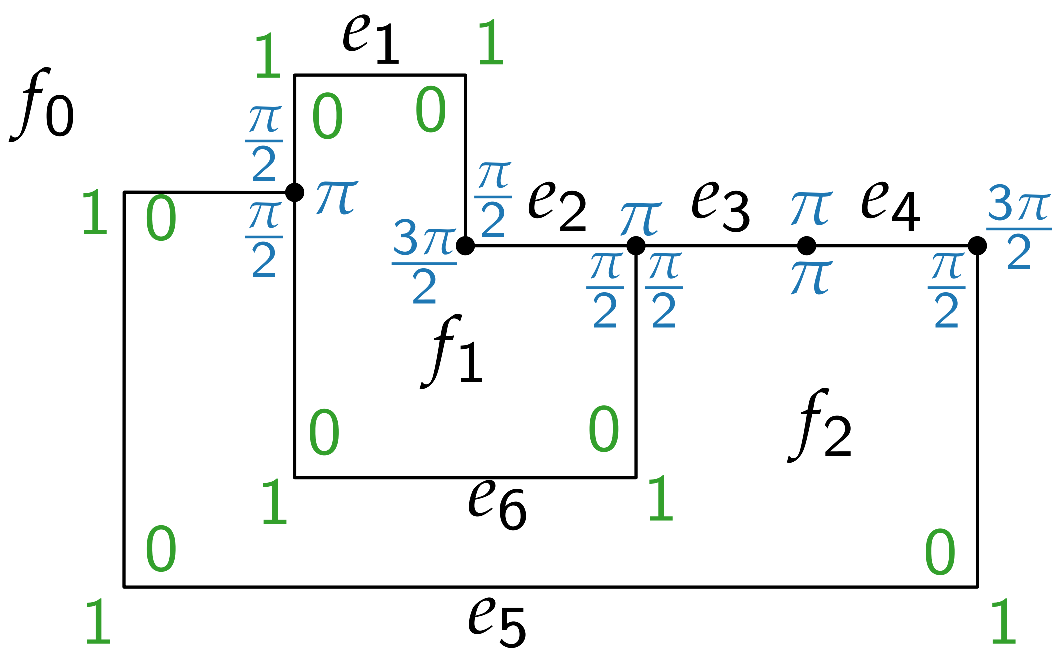

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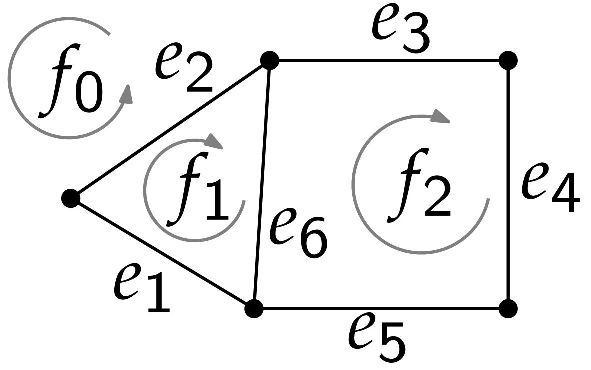
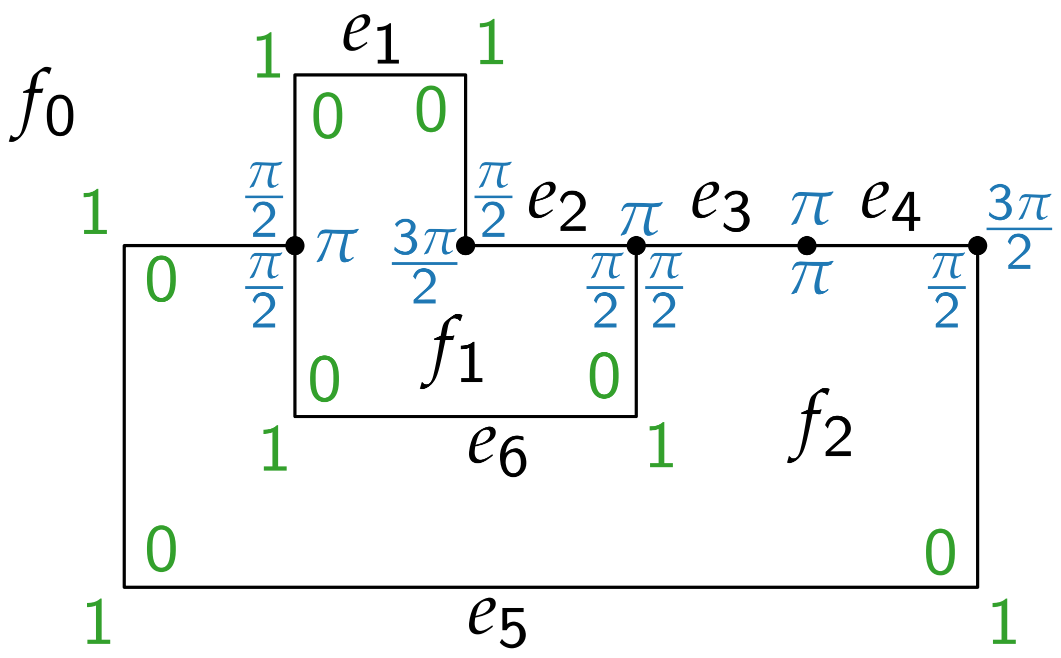
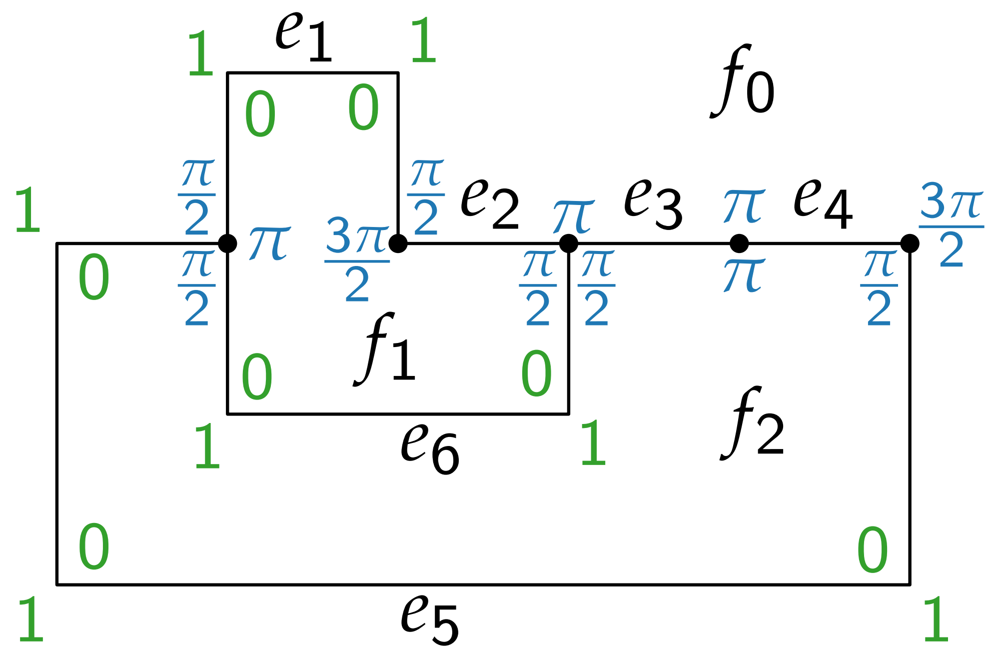




Concrete coordinates are not fixed yet!

Correctness of an orthogonal representation

(H1) $H(G)$ corresponds to F, f_0.



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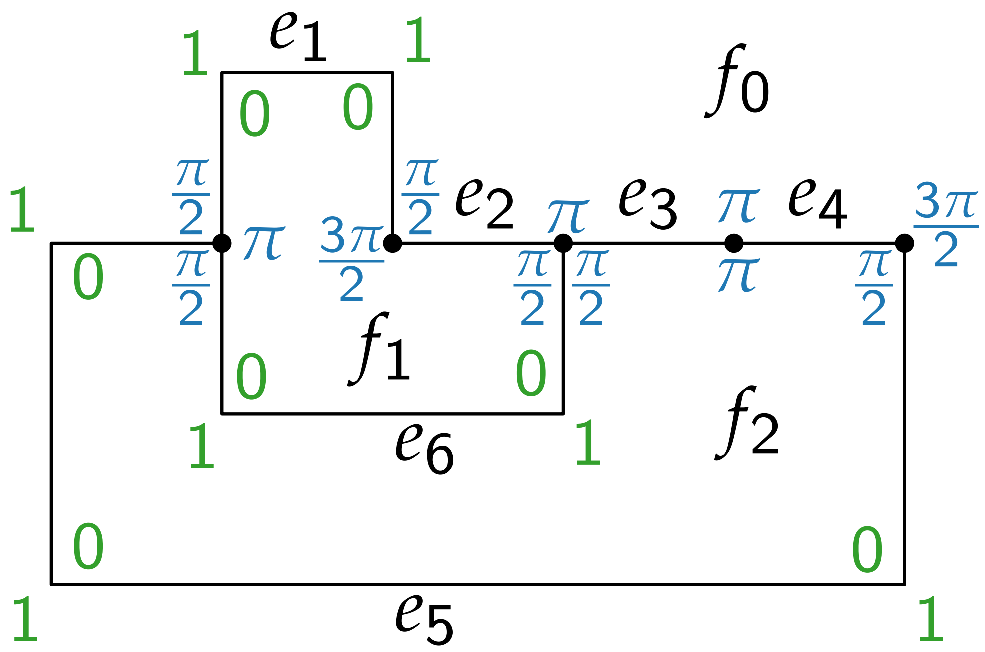
(H1) $H(G)$ corresponds to F, f_0.

(H2) For an edge $\{u, v\}$ shared by faces f and g with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$ sequence δ_1 is reversed and inverted δ_2.



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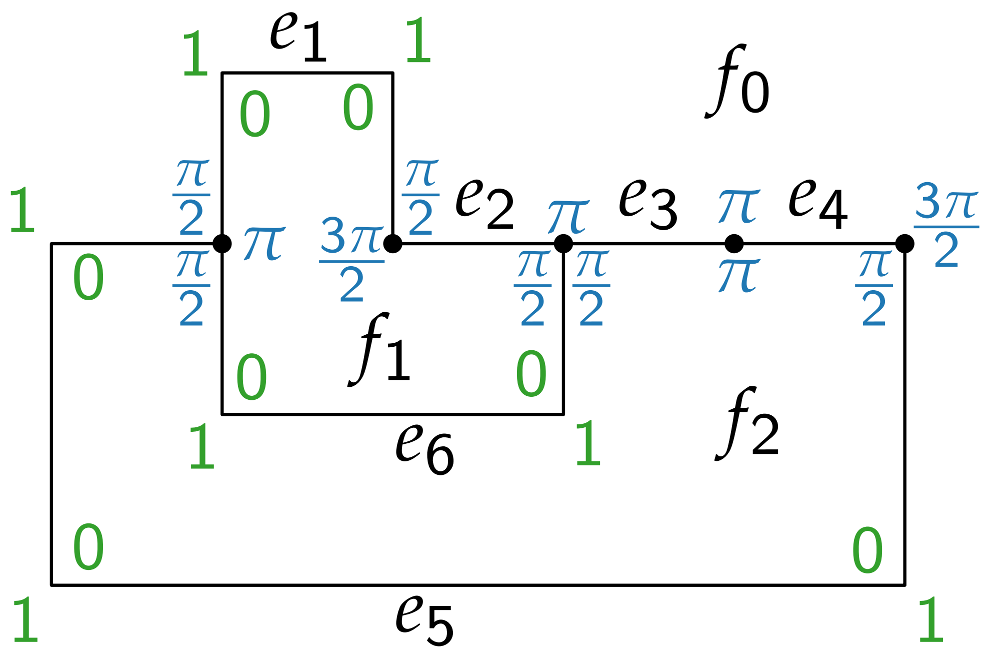
(H3) Let $|\delta|_0$ (resp. $|\delta|_1$) be the number of zeros (resp. ones) in δ and $r = (e, \delta, \alpha)$.
For $C(r) := |\delta|_0 - |\delta|_1 + 2 - 2\alpha/\pi$ it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

$C(r)$: The "total turn" (in units of $\frac{\pi}{2}$) of e in f



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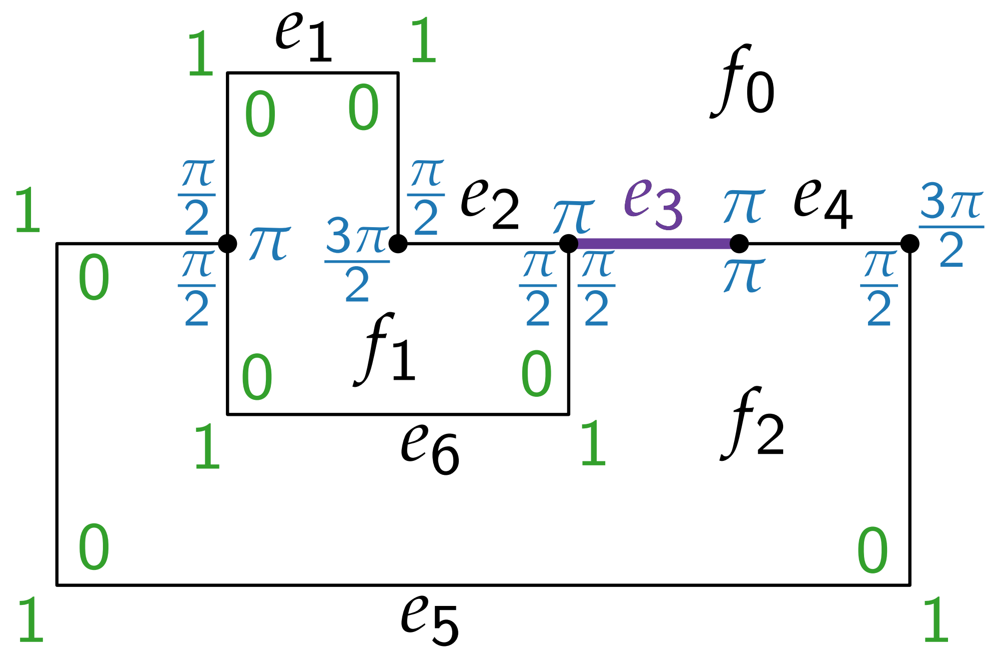

$$C(e_3) = 0 - 0 + 2 - \frac{2\pi}{\pi} = 0$$

Correctness of an orthogonal representation

(H1) $H(G)$ corresponds to F, f_0.

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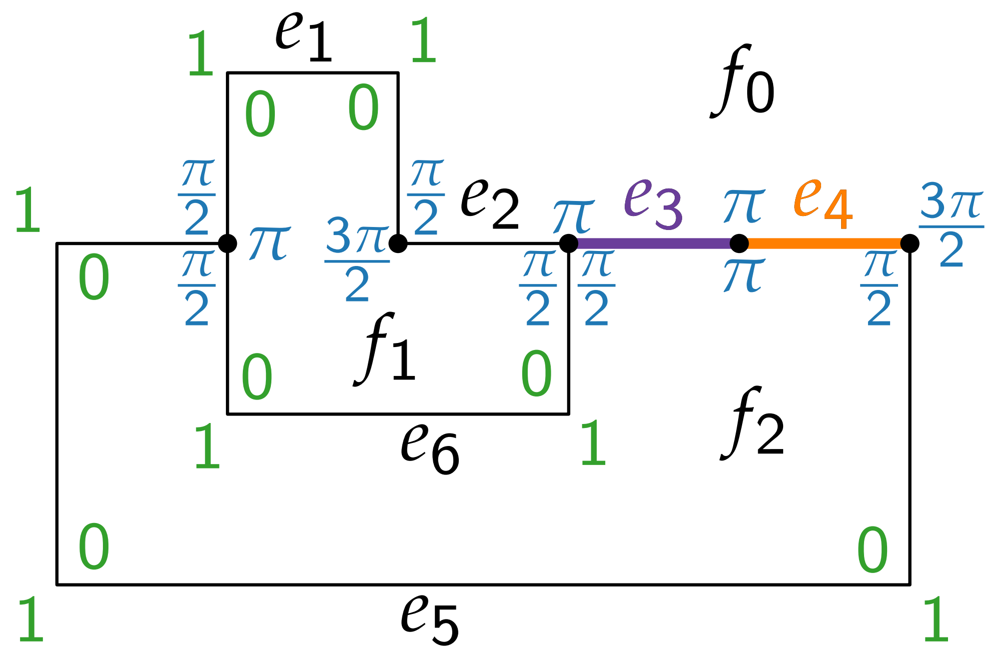
For $C(r) := |\delta|_0 - |\delta|_1 + 2 - 2\alpha/\pi$ it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

$C(r)$: The "total turn" (in units of $\frac{\pi}{2}$) of e in f



$$C(e_3) = 0 - 0 + 2 - \frac{2\pi}{\pi} = 0$$

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Correctness of an orthogonal representation

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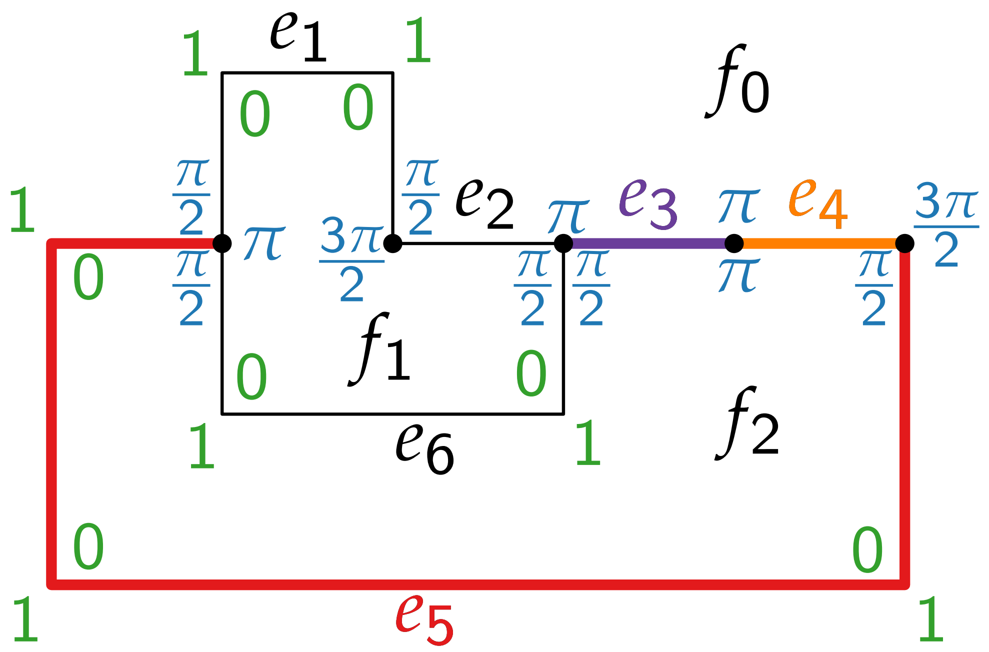

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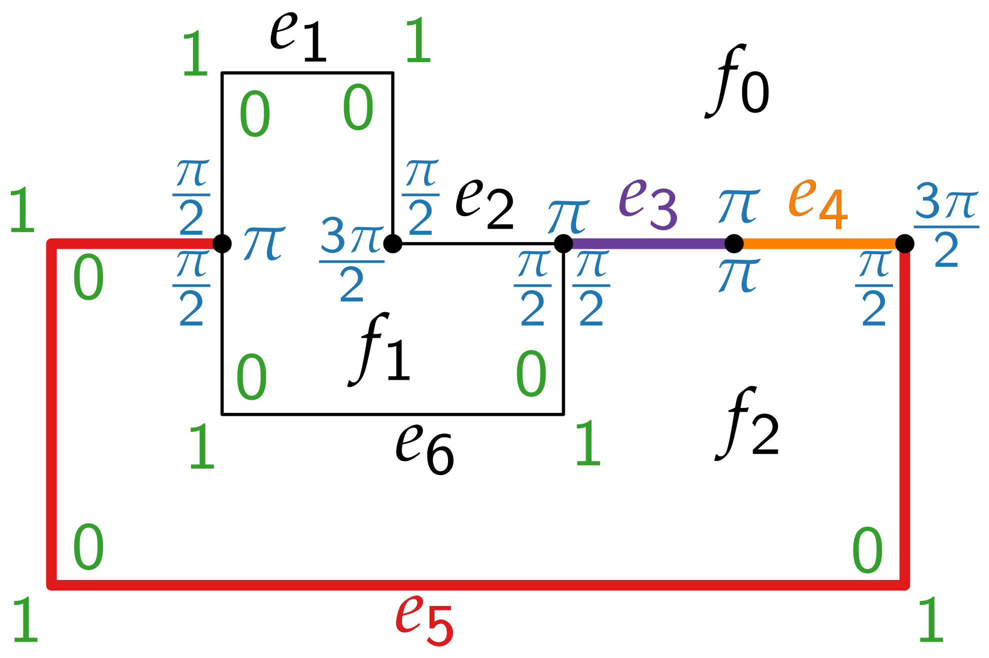
(H4) For each vertex v the sum of incident angles is 2π.



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Bend minimisation with given embedding

Geometric bend minimisation.

Given:
- Plane graph $G = (V, E)$ with maximum degree 4
- Combinatorial embedding F and outer face f_0

Find: Orthogonal drawing with minimum number of bends that preserves the embedding.

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Compare with the following variation.

Combinatorial bend minimisation.

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Idea.

Formulate as a network flow problem:

- a unit of flow $= \measuredangle \frac{\pi}{2}$
- vertices $\xrightarrow{\measuredangle}$ faces ($\# \measuredangle \frac{\pi}{2}$ per face)
- faces $\xrightarrow{\measuredangle}$ neighbouring faces ($\#$ bends toward the neighbour)

Reminder: s-t flow network

Flow network $(D = (V, A); s, t; u)$ with

- directed graph $D = (V, A)$
- edge *capacity* $u\colon A \to \mathbb{R}_0^+$
- *source* $s \in V$, *sink* $t \in V$

A function $\phi\colon A \to \mathbb{R}_0^+$ is called ***s-t*-flow**, if:

$$0 \leq \phi(i,j) \leq u(i,j) \qquad \forall (i,j) \in A \tag{1}$$

$$\sum_{(i,j)\in A} \phi(i,j) - \sum_{(j,i)\in A} \phi(j,i) = 0 \qquad \forall i \in V \setminus \{s,t\} \tag{2}$$

Reminder: general flow network

Flow network $(D = (V, A); \ell; u; b)$ with

- directed graph $D = (V, A)$
- edge *lower bound* $\ell \colon A \to \mathbb{R}_0^+$
- edge *capacity* $u \colon A \to \mathbb{R}_0^+$
- node *production/consumption* $b \colon V \to \mathbb{R}$ with $\sum_{i \in V} b(i) = 0$

A function $\phi \colon A \to \mathbb{R}_0^+$ is called **valid flow**, if:

$$\ell(i,j) \leq \phi(i,j) \leq u(i,j) \qquad \forall (i,j) \in A \tag{3}$$

$$\sum_{(i,j) \in A} \phi(i,j) - \sum_{(j,i) \in A} \phi(j,i) = b(i) \qquad \forall i \in V \tag{4}$$

Problems for flow networks

Valid flow problem.
Find a valid flow $\phi \colon A \to \mathbb{R}_0^+$, i.e., such that

- lower bounds $\ell(e)$ and capacities $u(e)$ are respected (inequalities (3)) and
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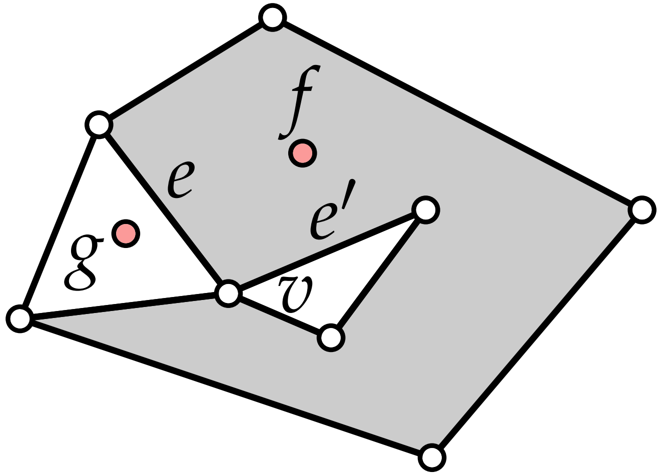
Miminum cost flow problem.
Find a valid flow $\phi\colon A \to \mathbb{R}_0^+$, that minimises cost function $\text{cost}(\phi)$ (over all valid flows).

Flow network for bend minimisation

Define flow network $N(G) = ((V \cup F, A); \ell; u; b; \text{cost})$:



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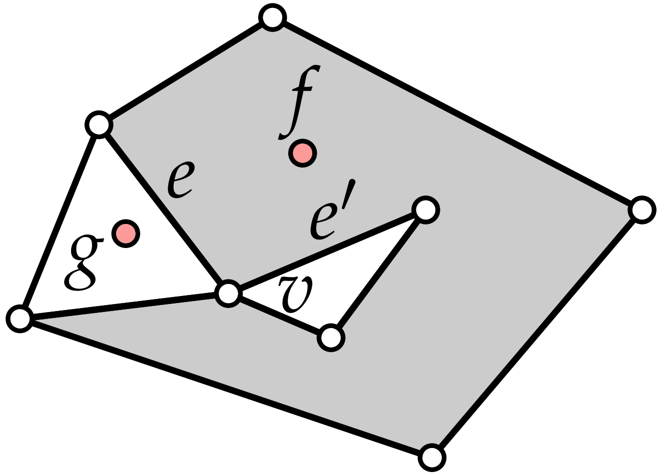

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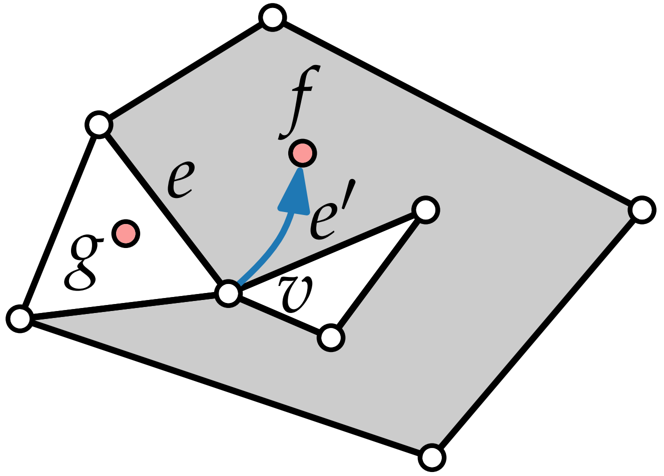

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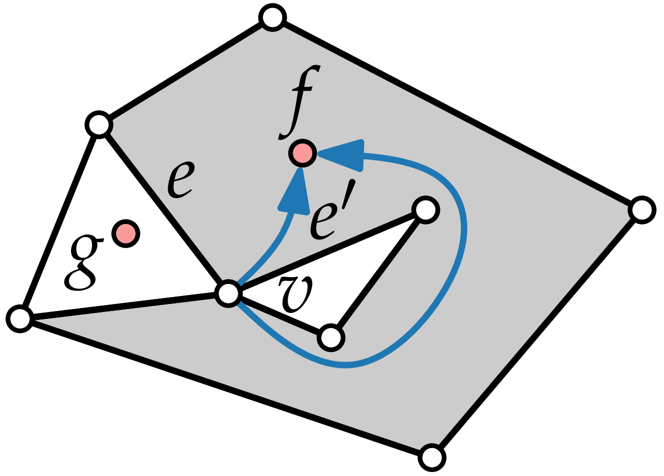

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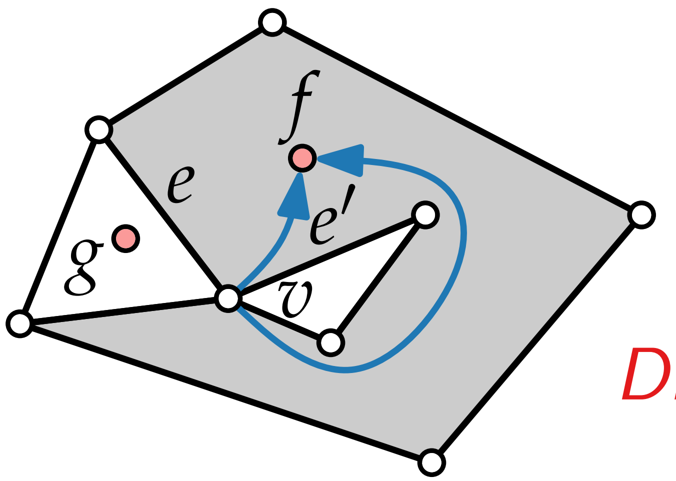

Directed multigraph!

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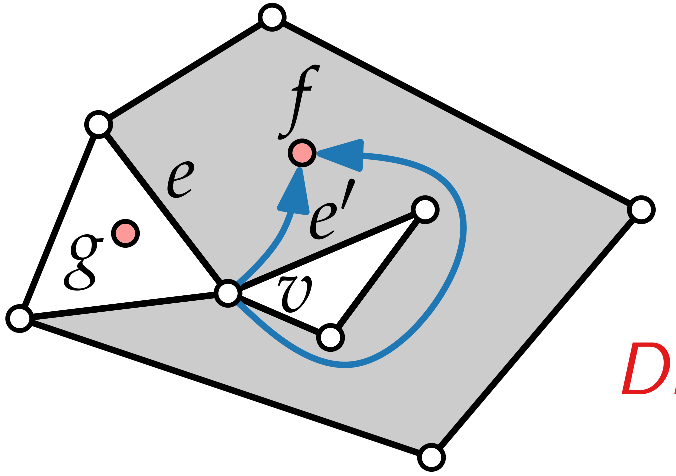

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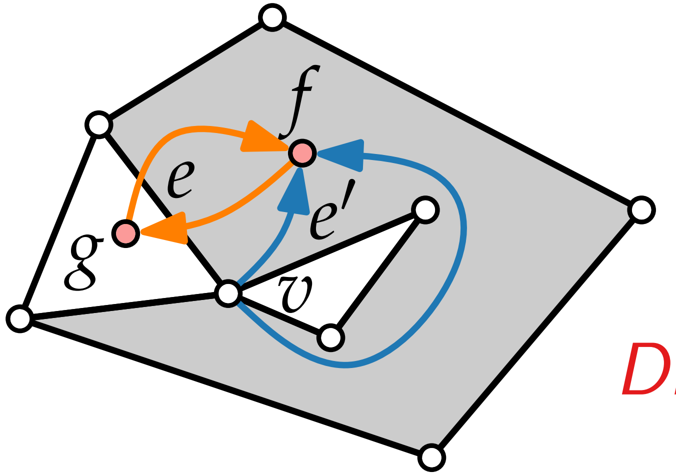

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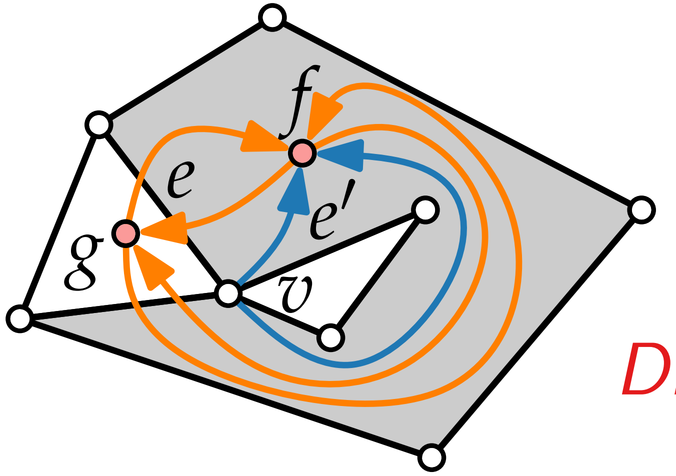

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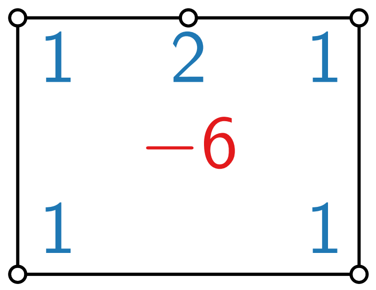

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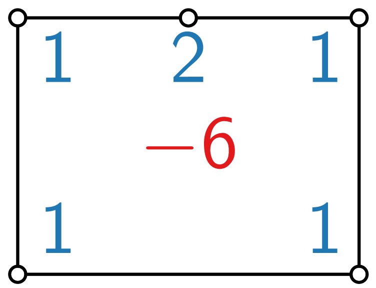

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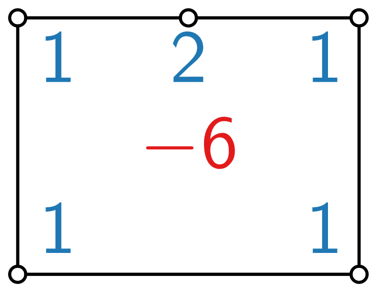

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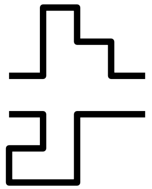
We model only the *number* of bends. Why is it enough?

→ Exercise



Constraint on "production/consumption"

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Proof: Let $n = |V|,\ m = |E|,\ f = |F|$

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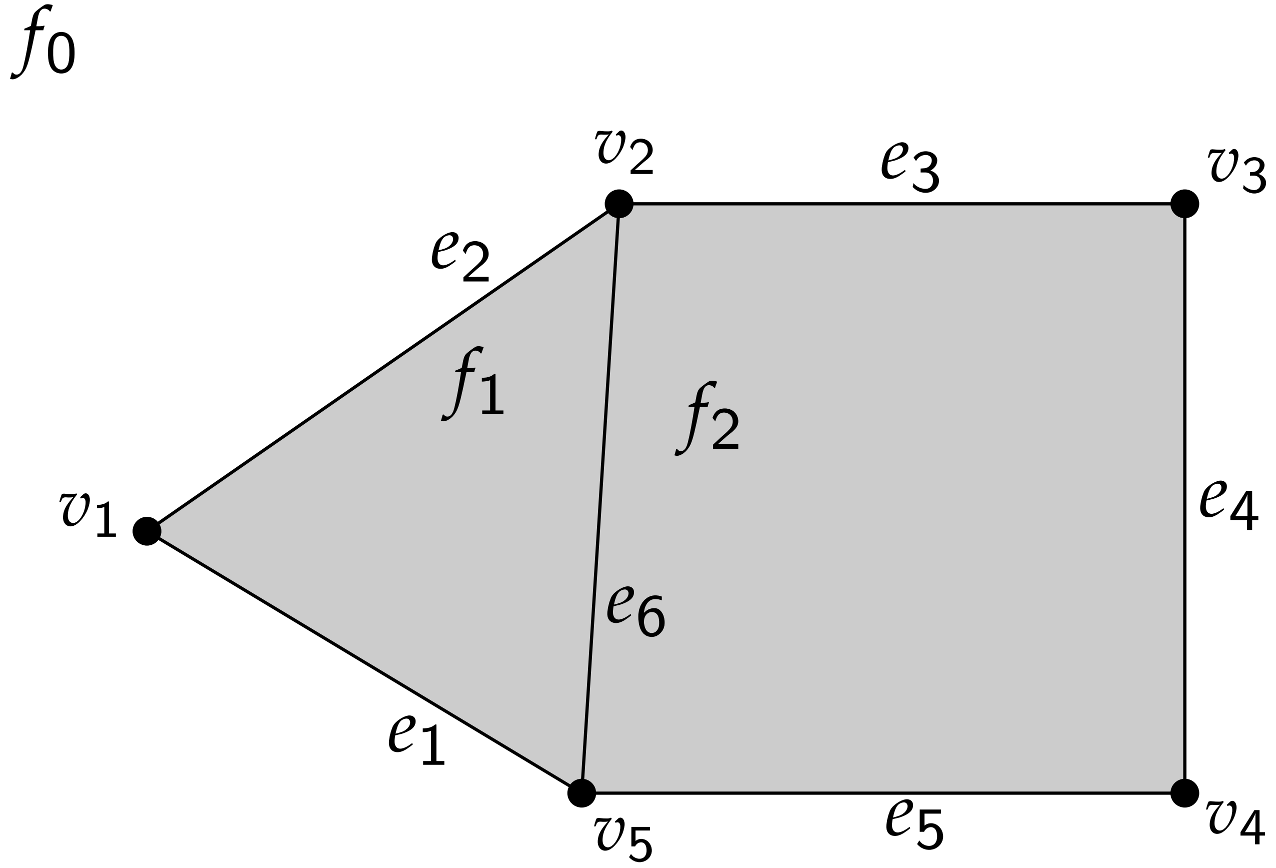
Proof: Let $n = |V|,\ m = |E|,\ f = |F|$

$$\begin{aligned} \textstyle\sum_w b(w) &= \textstyle\sum_{v \in V} b(v) + \sum_{f \in F} b(f) \\ &= 4n + \sum_{f \in F \setminus \{f_0\}} (-2\deg_G(f) + 4) \ + \ (-2\deg_G(f_0) - 4 \\ &= 4n - 2\sum_{f \in F} \deg_G(f) + 4(f-1) - 4 \\ &= 4n - 2 \cdot 2m + 4f - 8 \ = \ 4(n - m + f - 2) \overset{Euler}{=} 0 \quad \blacksquare \end{aligned}$$

Flow network example



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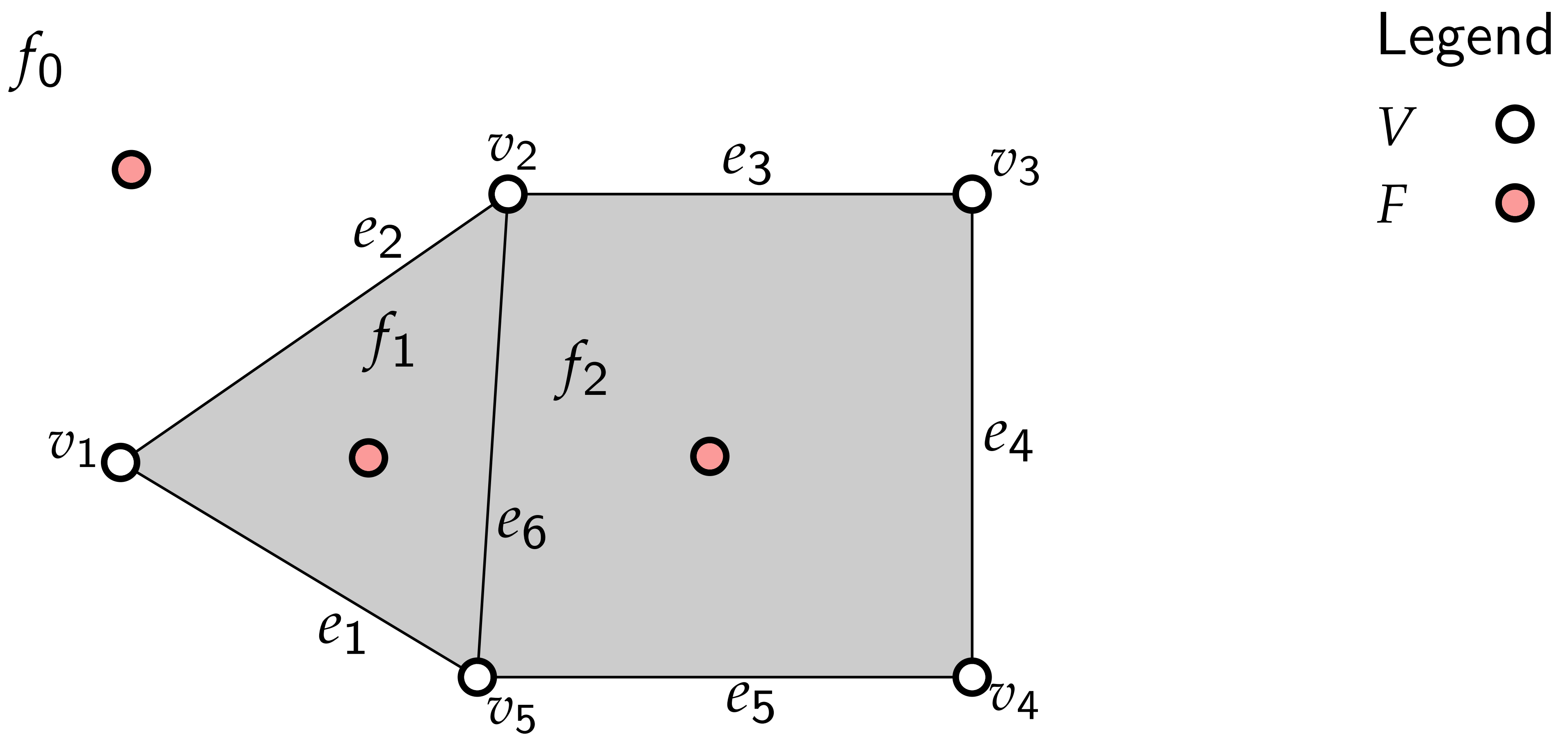

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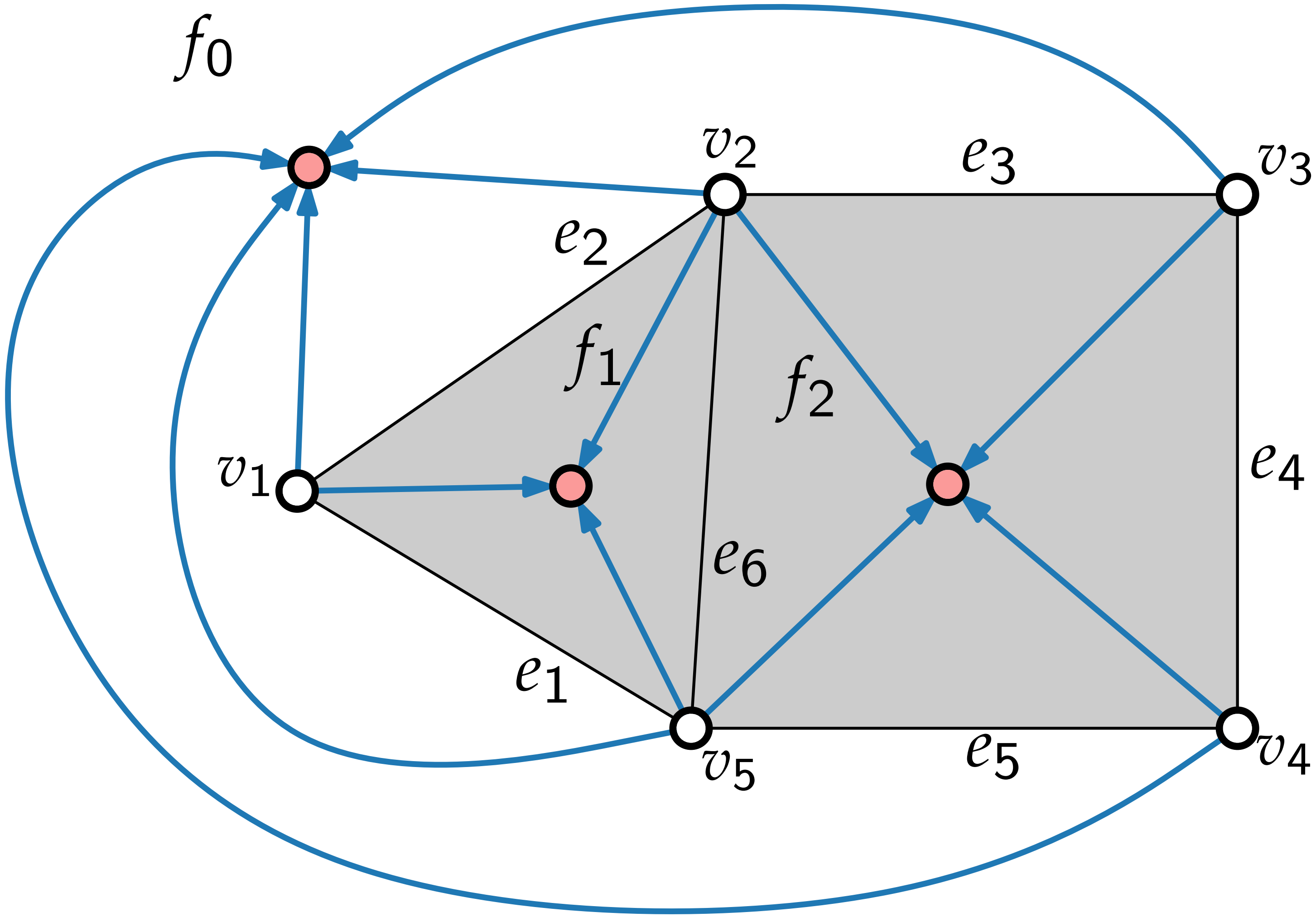
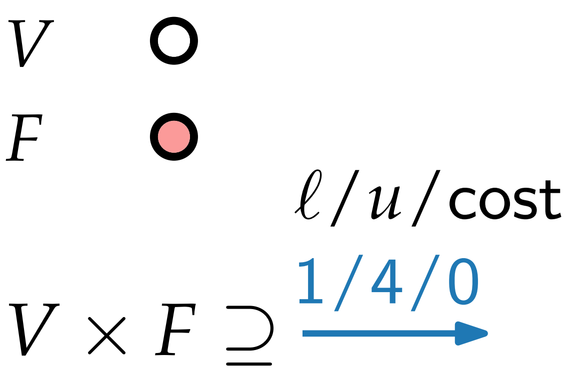


Legend



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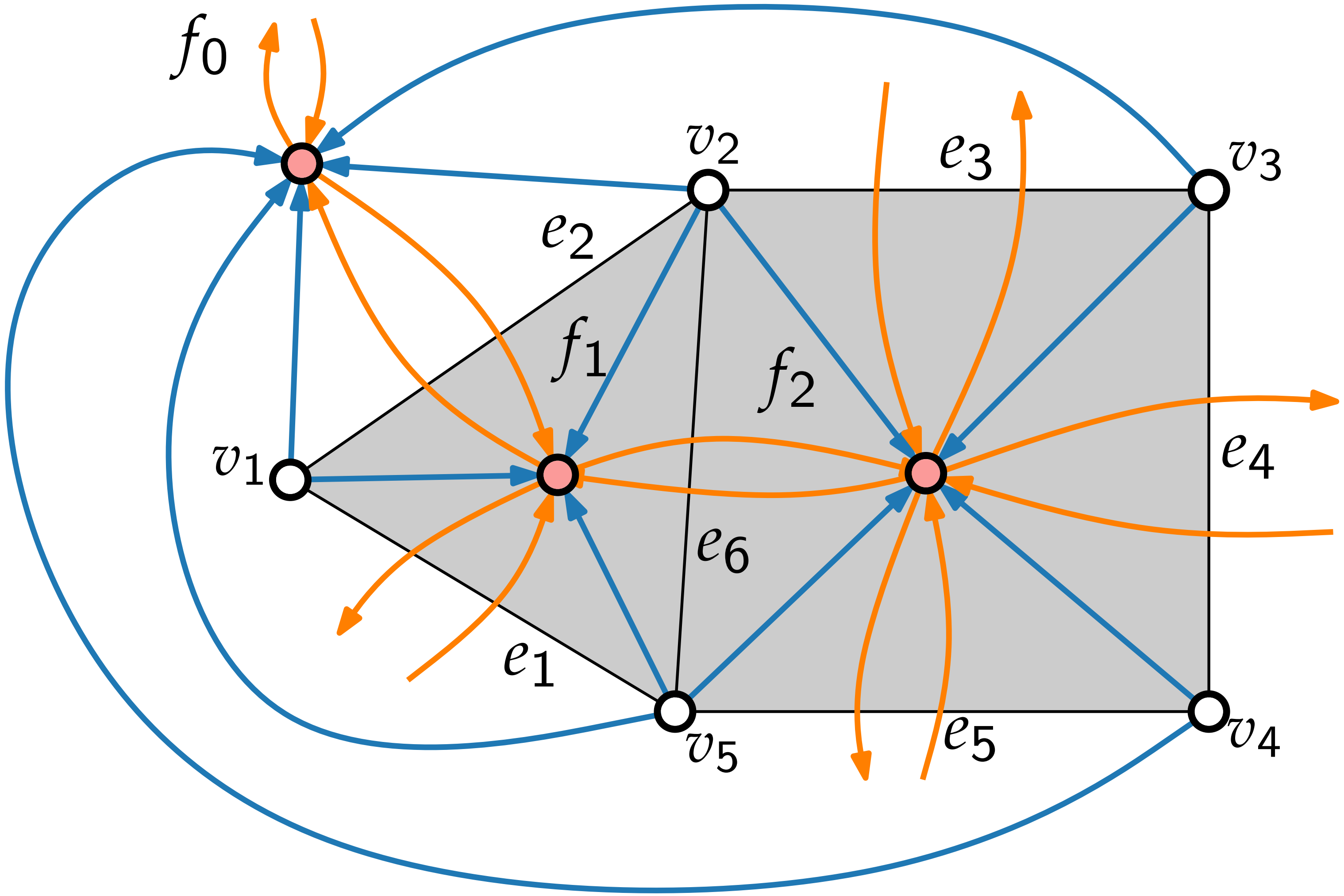


Legend

V ○

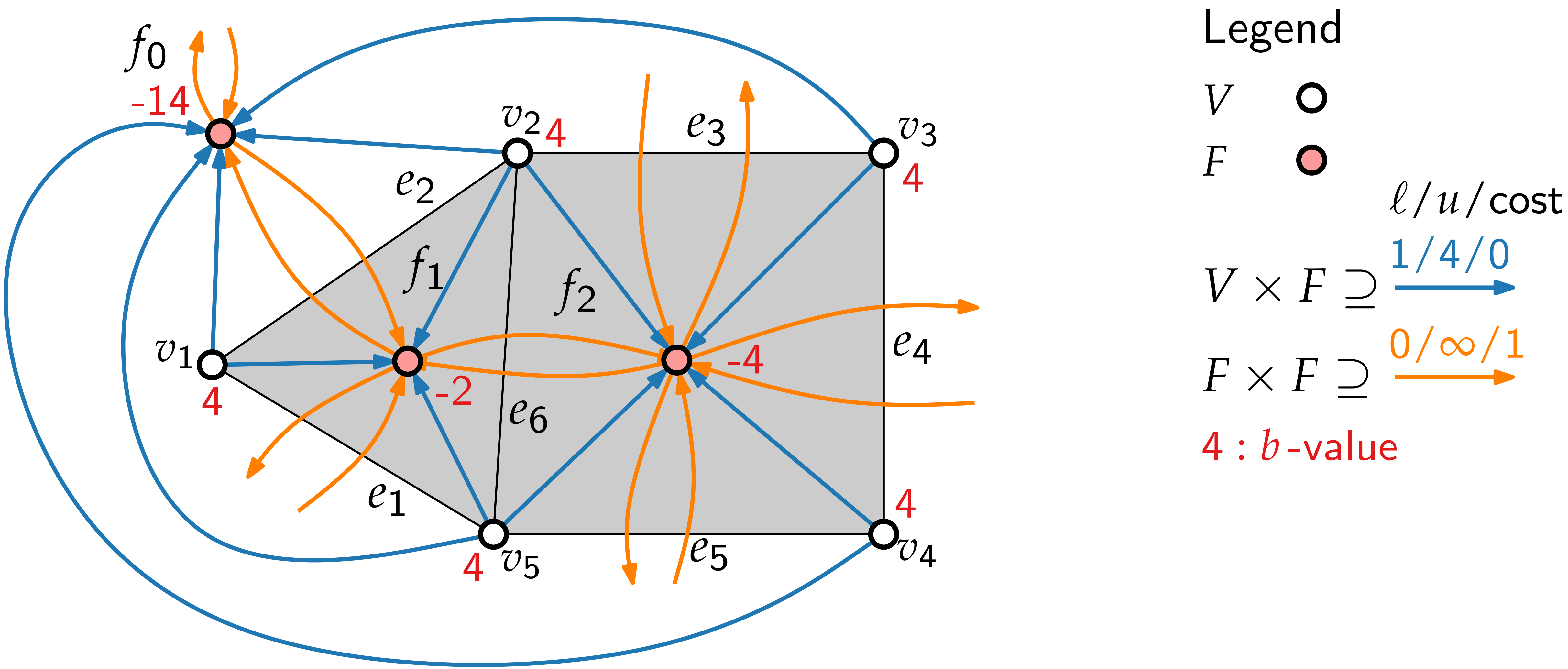
F ●

$\ell\,/\,u\,/\,$cost

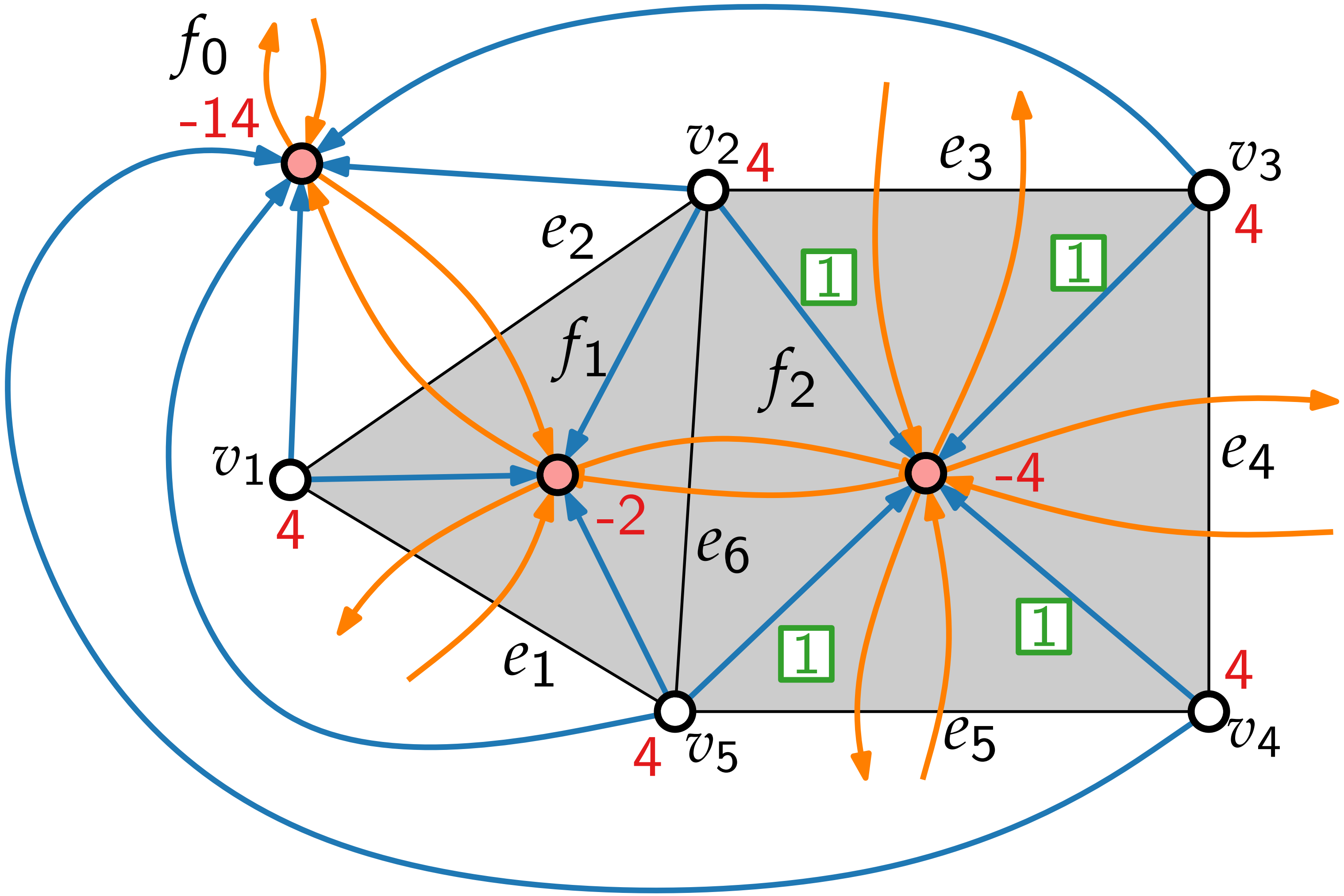
$V \times F \supseteq$ 1/4/0 →

$F \times F \supseteq$ 0/∞/1 →

Flow network example



Flow network example



Legend

V ○

F ●

$\ell / u /$ cost

$V \times F \supseteq$ 1/4/0

$F \times F \supseteq$ 0/∞/1

4 : b-value

3 flow

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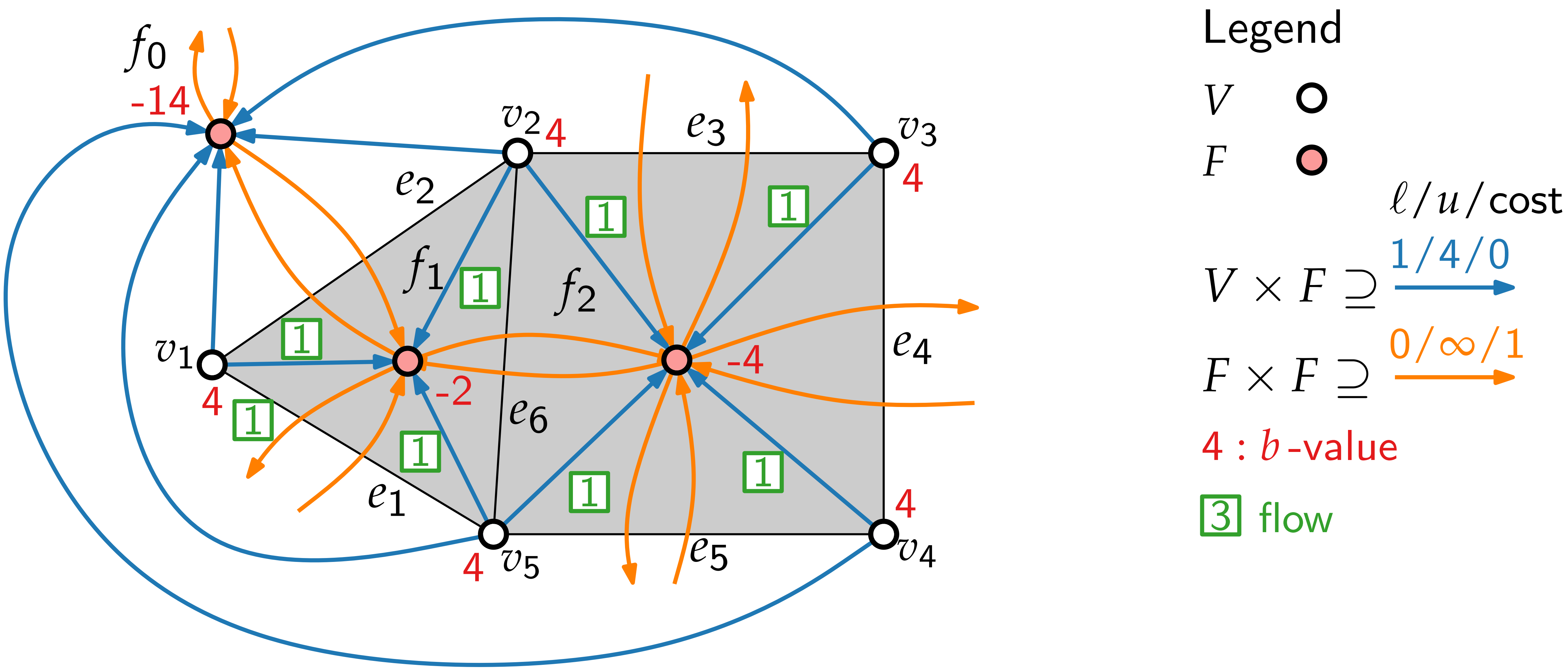

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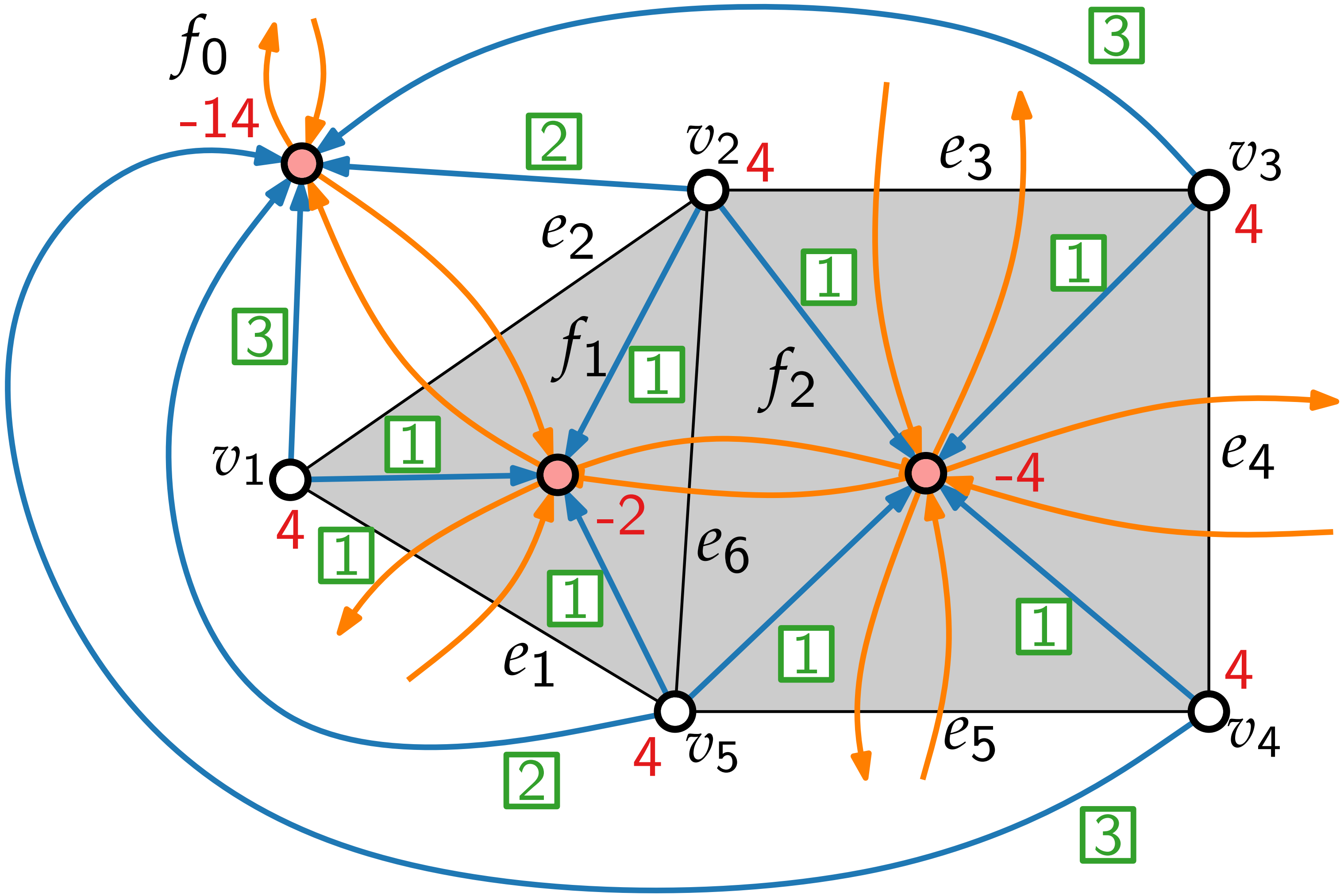

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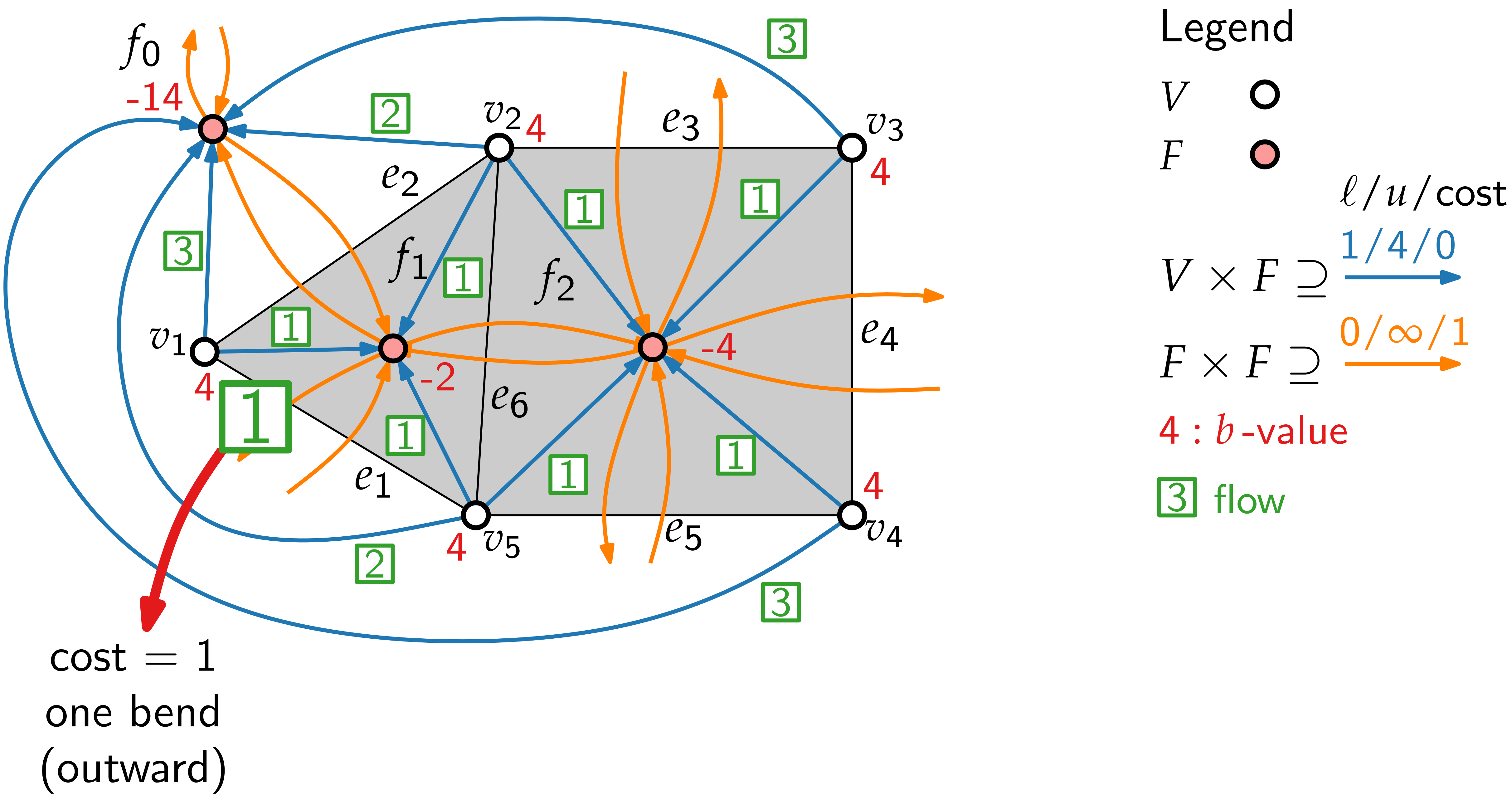

Bend minimisation – result

Theorem. [Tamassia '87]
A plane graph (G, F, f_0) has a valid orthogonal representation $H(G)$ with k bends iff the flow network $N(G)$ has a valid flow ϕ of cost k.

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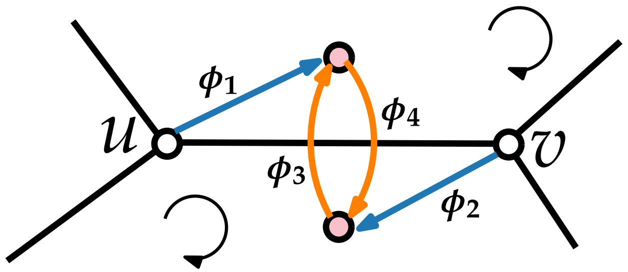

$$\Rightarrow \begin{cases} \left((u,v), 0^{\phi_3} 1^{\phi_4}, \phi_2 \frac{\pi}{2}\right) \\ \left((v,u), 0^{\phi_4} 1^{\phi_3}, \phi_1 \frac{\pi}{2}\right) \end{cases}$$

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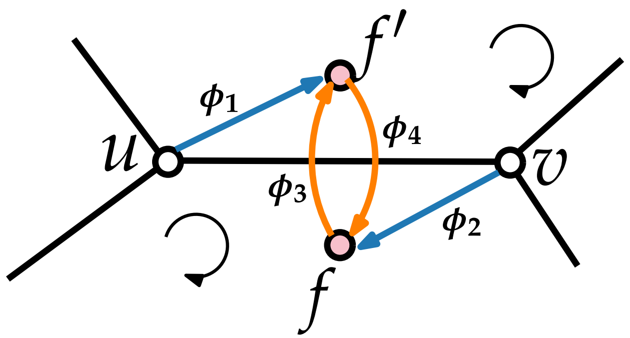

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\sum_{r \in H(f)} C(r) &= \sum_{r \in H(f)} (2 + |\delta_r|_0 - |\delta_r|_1 - \frac{\alpha_r}{\pi/2}) \\
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(H4) Total angle at each vertex $= 2\pi$ ✓

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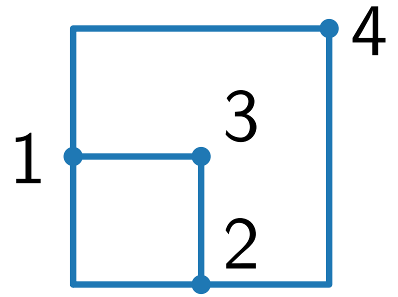
- This special flow problem for a planar network $N(G)$ can be solved in $O(n^{3/2})$ time. [Cornelsen, Karrenbauer GD 2011]
- Bend minimization without a given combinatorial embedding is an NP-hard problem. [Garg, Tamassia SIAM J. Comput. 2001]

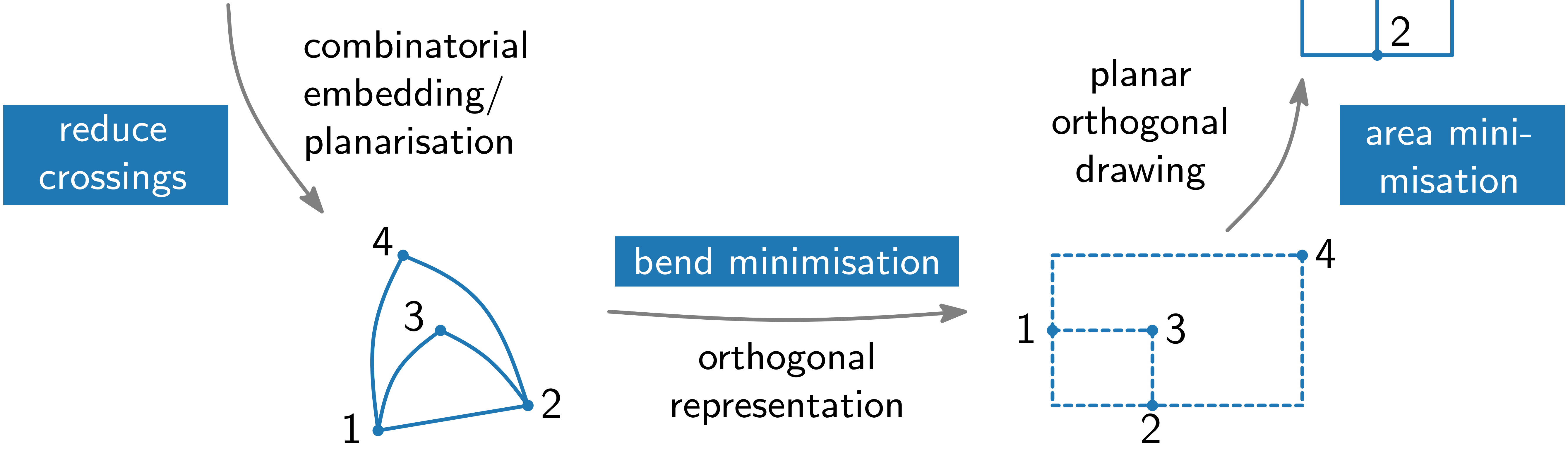
Topology - Shape - Metrics

Three-step approach:

[Tamassia SIAM J. Comput. 1987]

$V = \{v_1, v_2, v_3, v_4\}$
$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$





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Compaction problem.

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Idea.

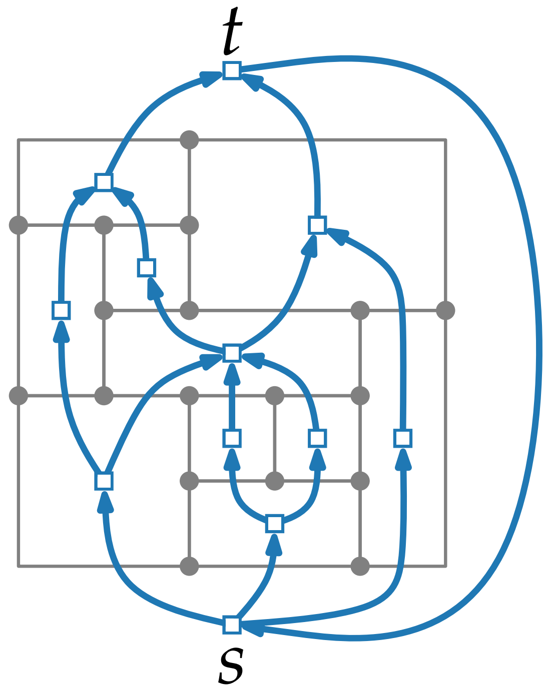
- Formulate flow network for horizontal/vertical compaction

Flow network for edge length assignment

Definition.

Flow Network $N_{\text{hor}} = ((W_{\text{hor}}, A_{\text{hor}}); \ell; u; b; \text{cost})$

- $W_{\text{hor}} = F \setminus \{f_0\} \cup \{s, t\}$ □
- $A_{\text{hor}} = \{(f, g) \mid f, g$ share a *horizontal* segment and f lies *below* $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in A_{\text{hor}}$
- $u(a) = \infty \quad \forall a \in A_{\text{hor}}$
- $\text{cost}(a) = 1 \quad \forall a \in A_{\text{hor}}$
- $b(f) = 0 \quad \forall f \in W_{\text{hor}}$



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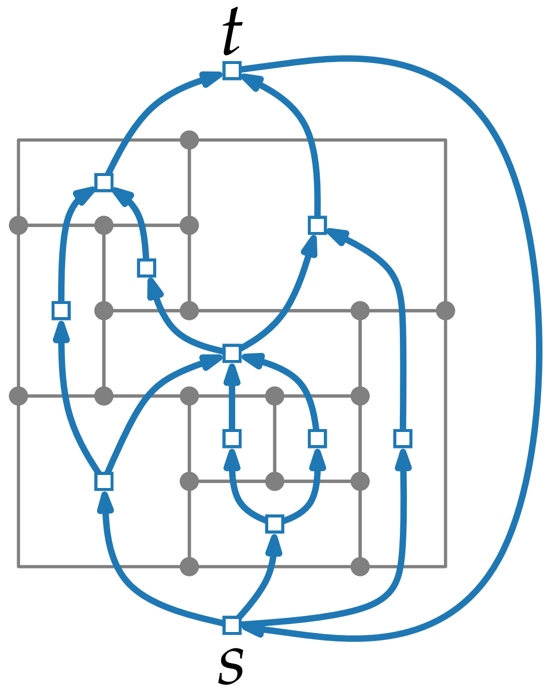

s and t represent lower and upper side of f_0

Flow network for edge length assignment

Definition.

Flow Network $N_{\text{ver}} = ((W_{\text{ver}}, A_{\text{ver}}); \ell; u; b; \text{cost})$

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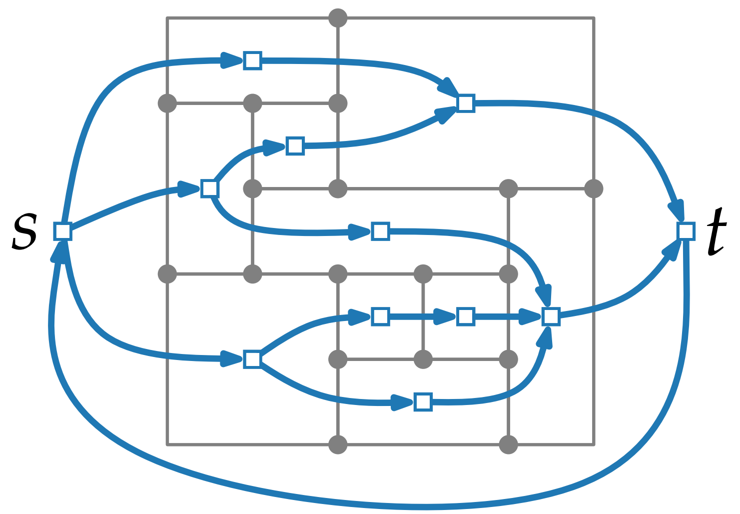
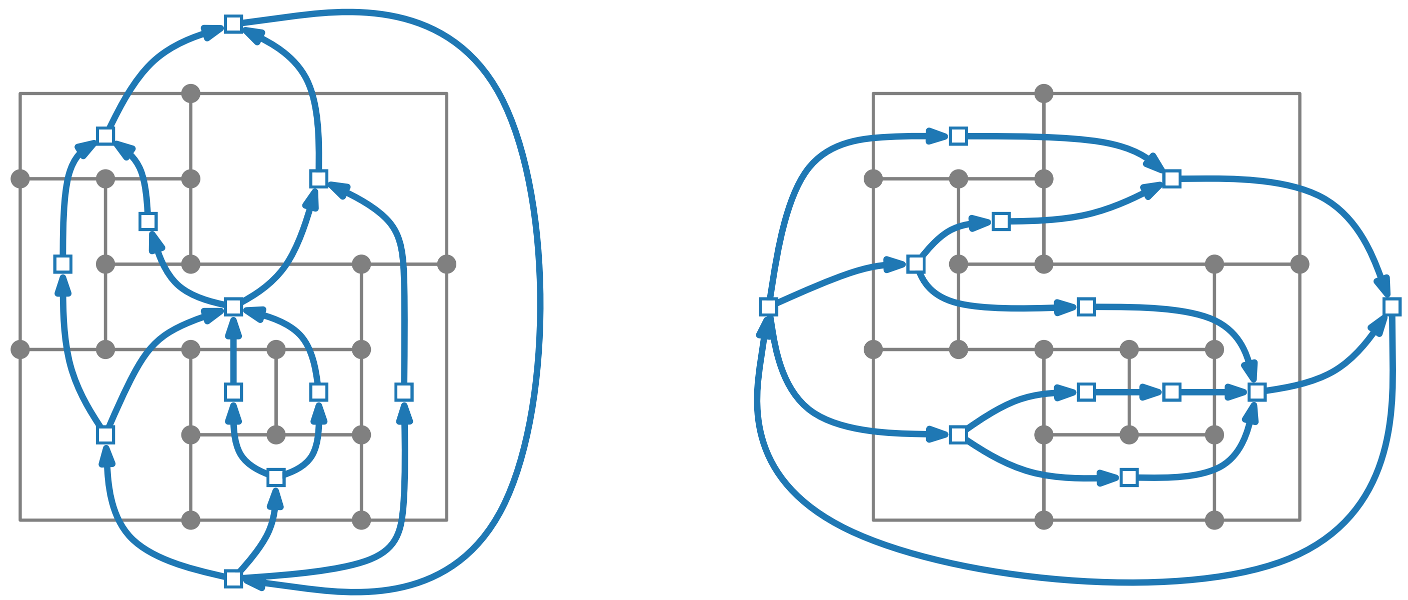


Compaction – result



Theorem.
Valid min-cost-flows for N_{hor} and N_{ver} exists iff corresponding edge lenghts induce orthogonal drawing.

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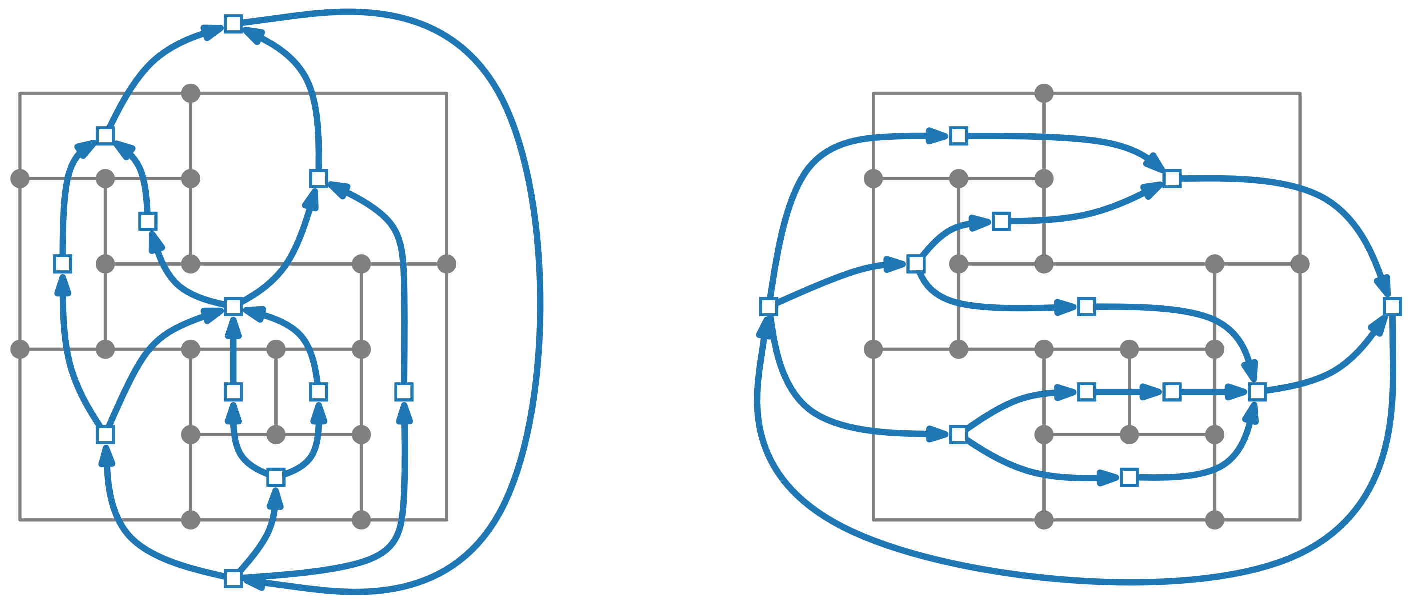

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What values of the drawing represent the following?

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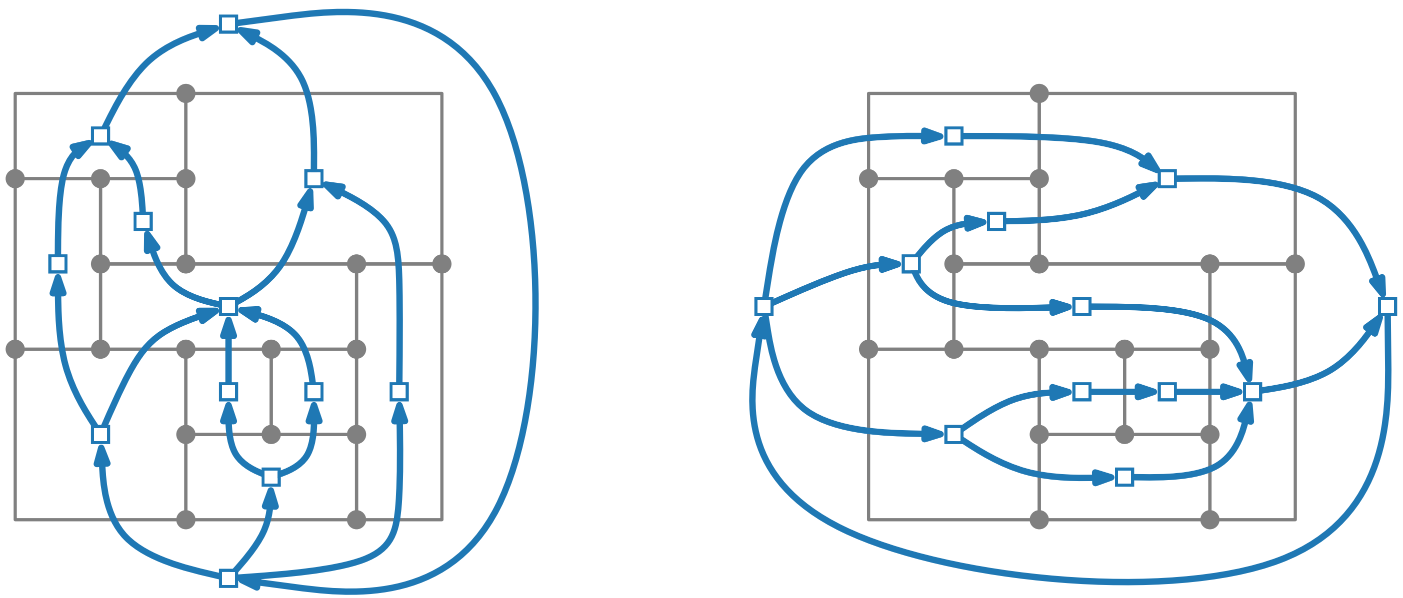

What if not all faces rectangular?

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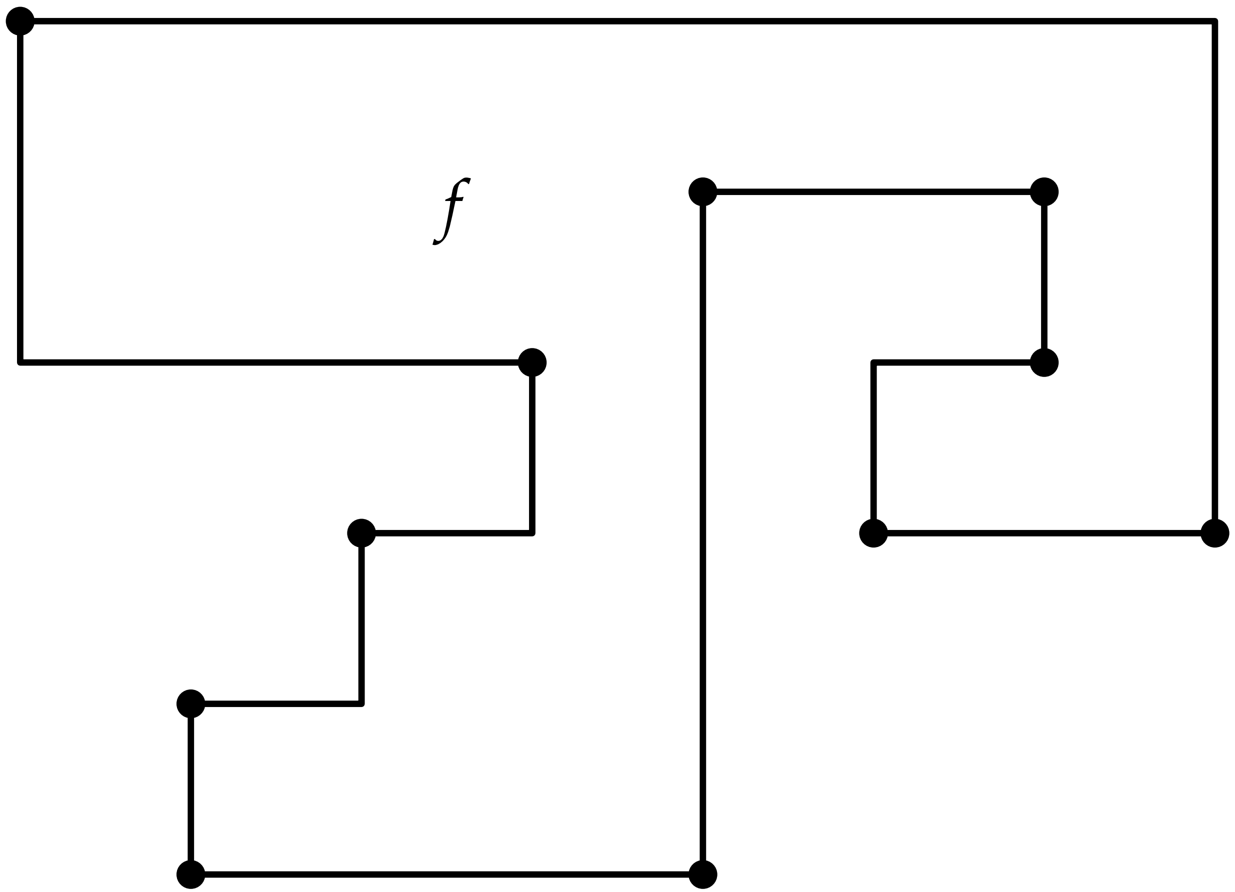
Refinement of (G, H) – inner face



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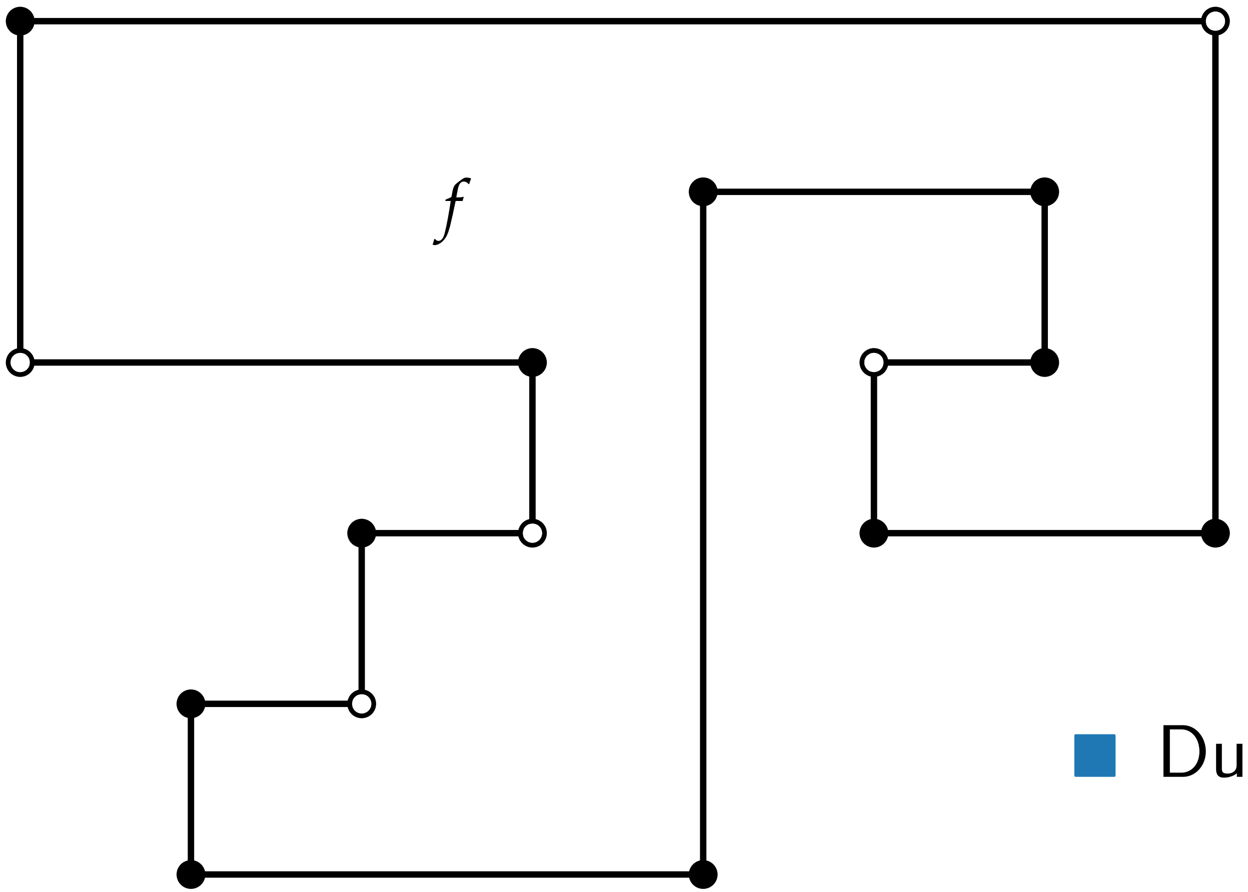

- Dummy vertices for bends

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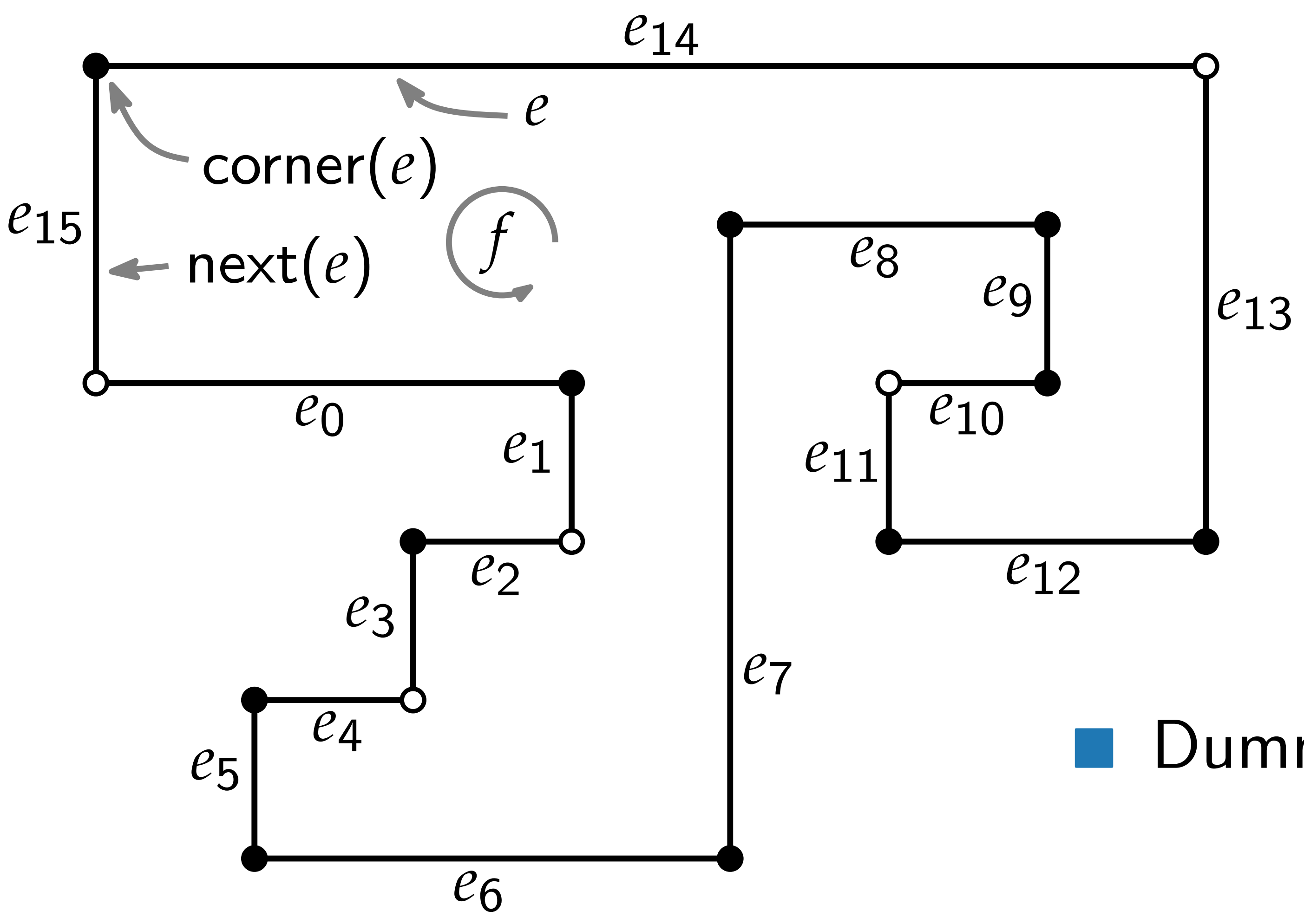

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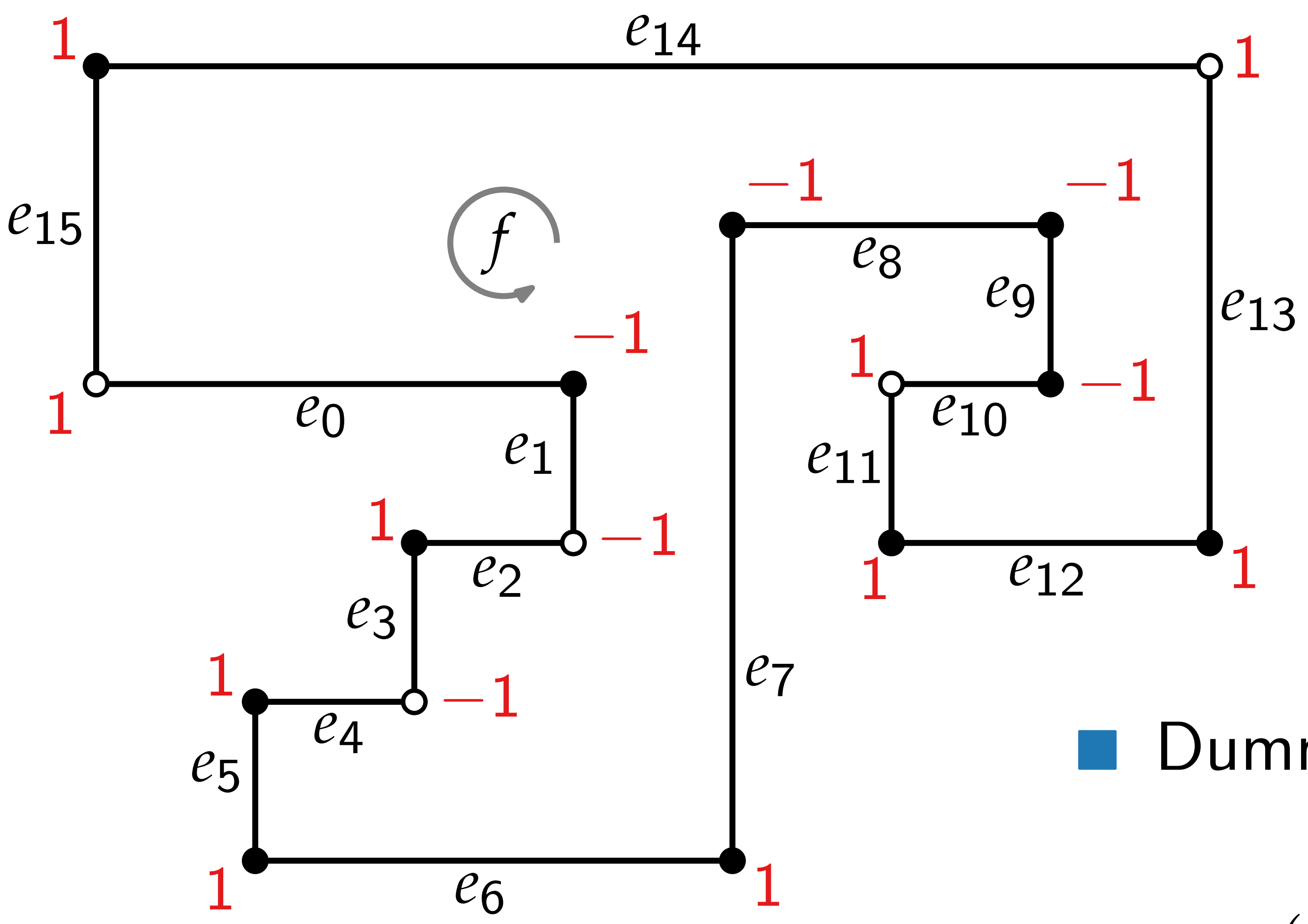

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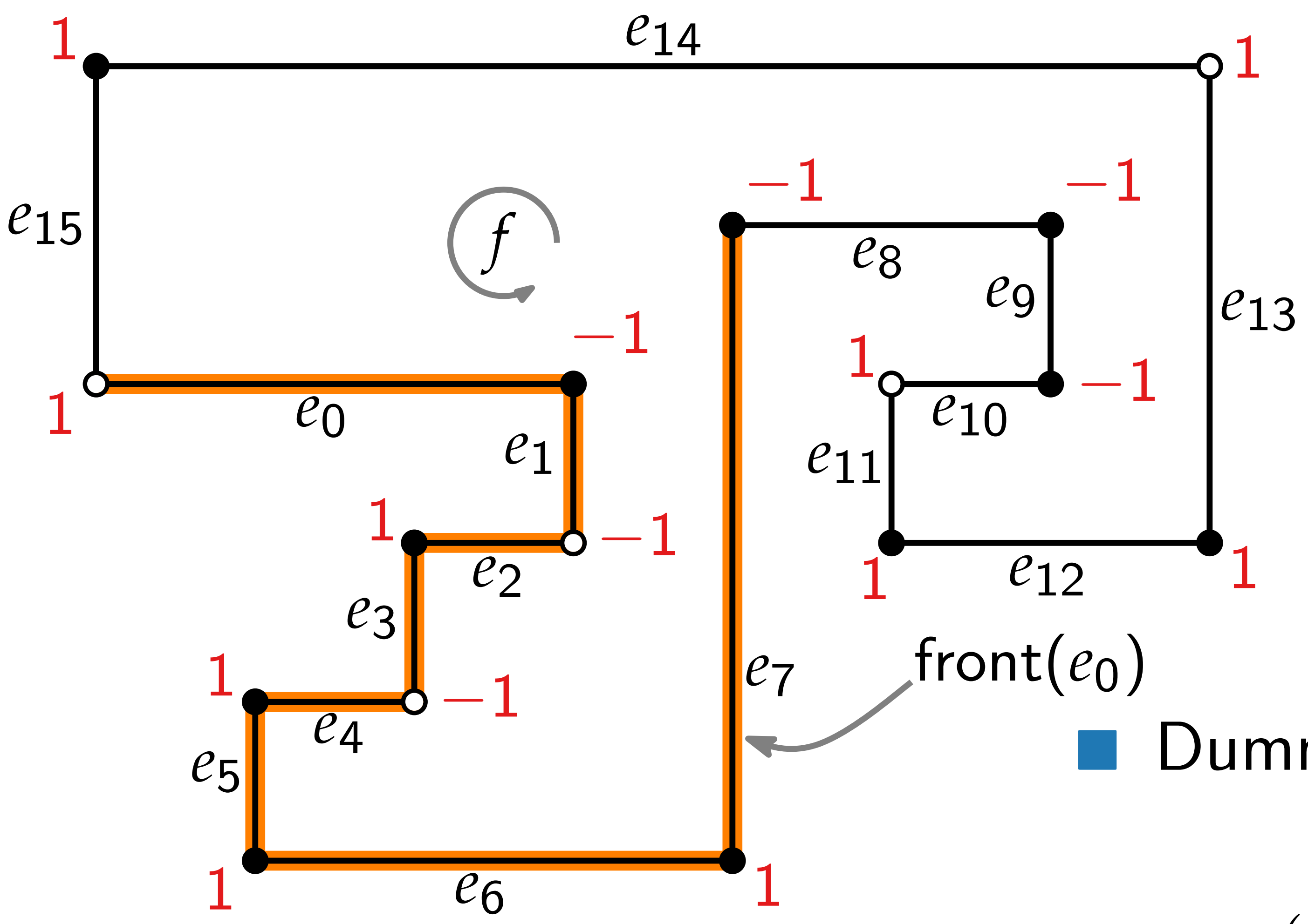

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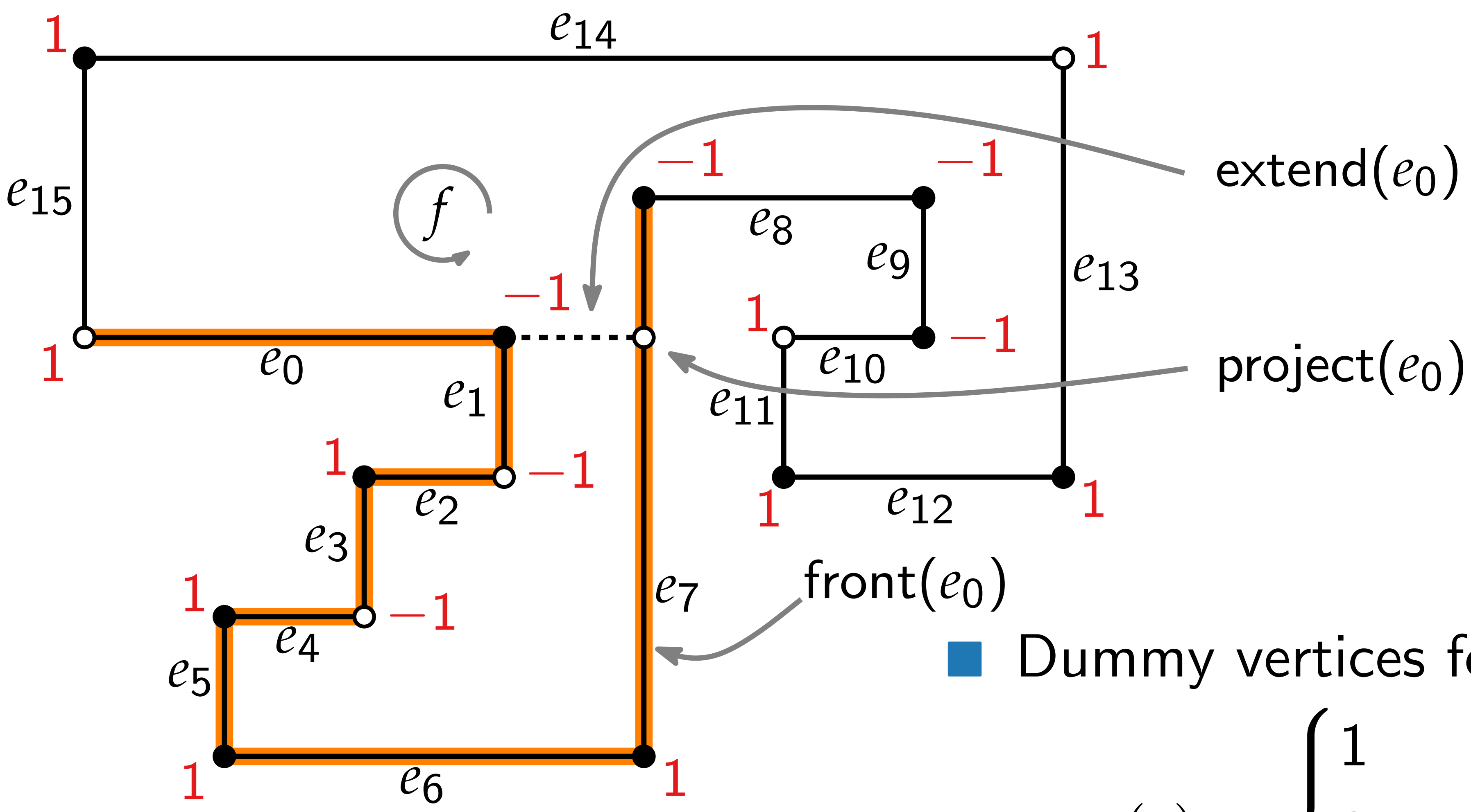

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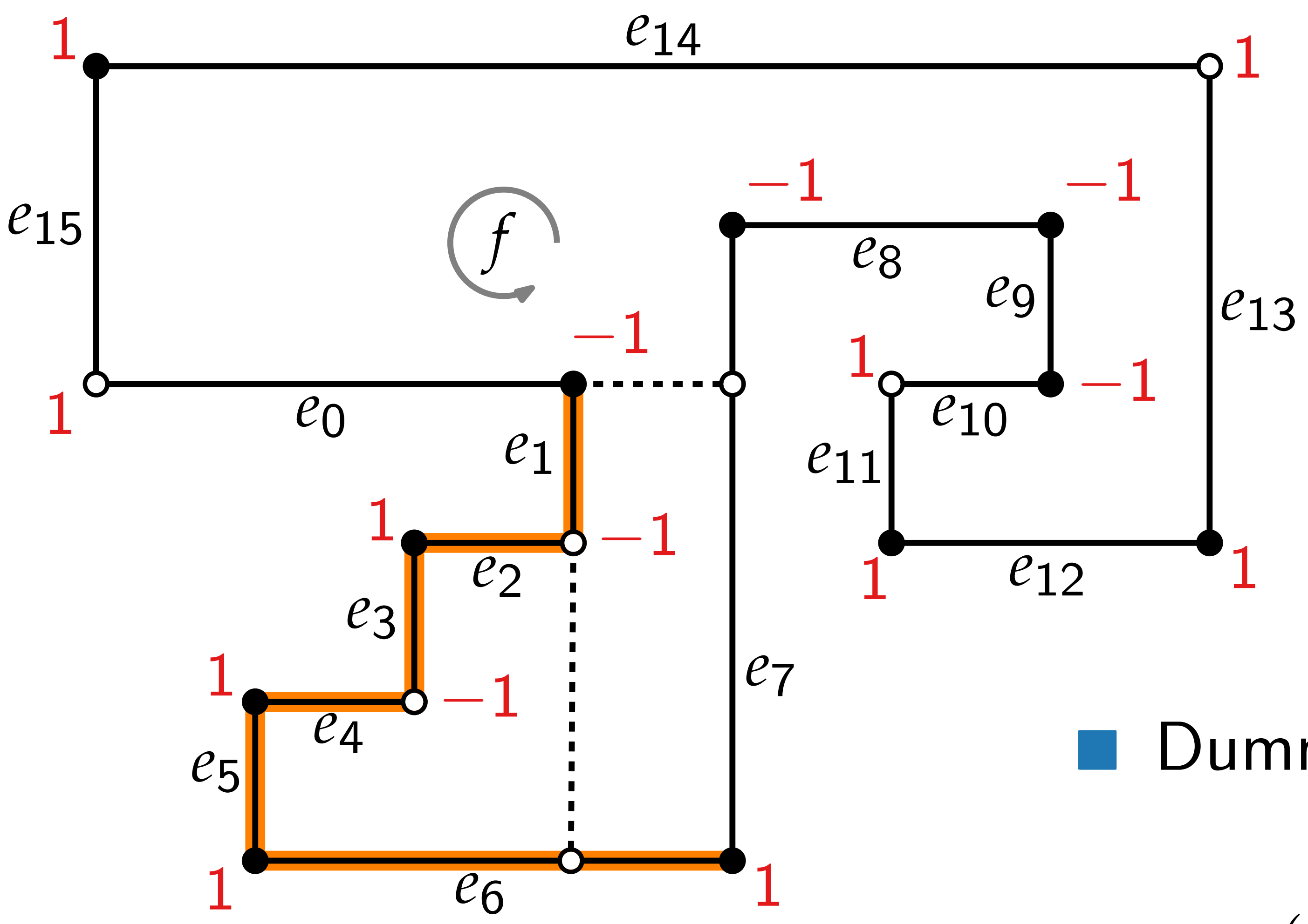

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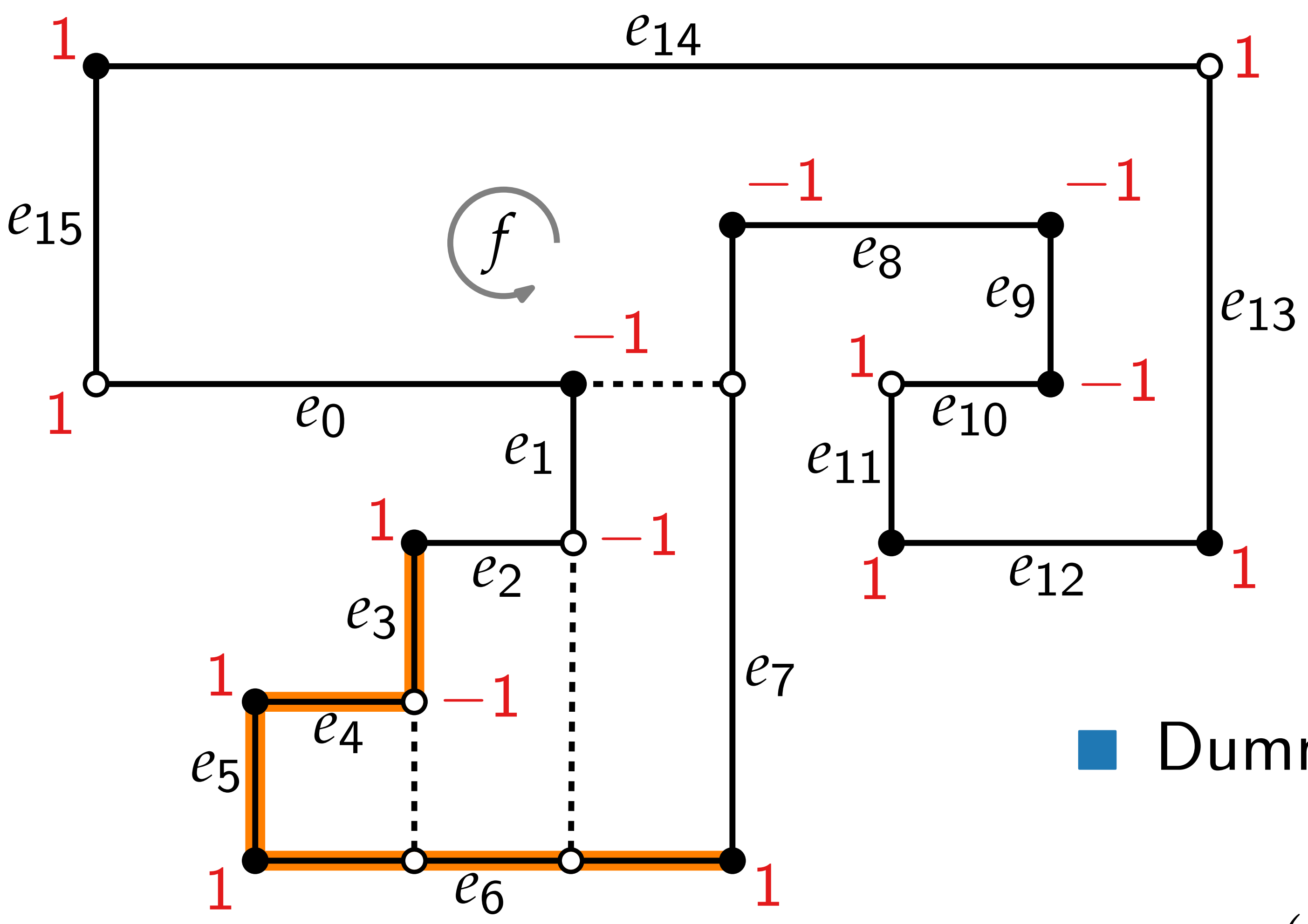

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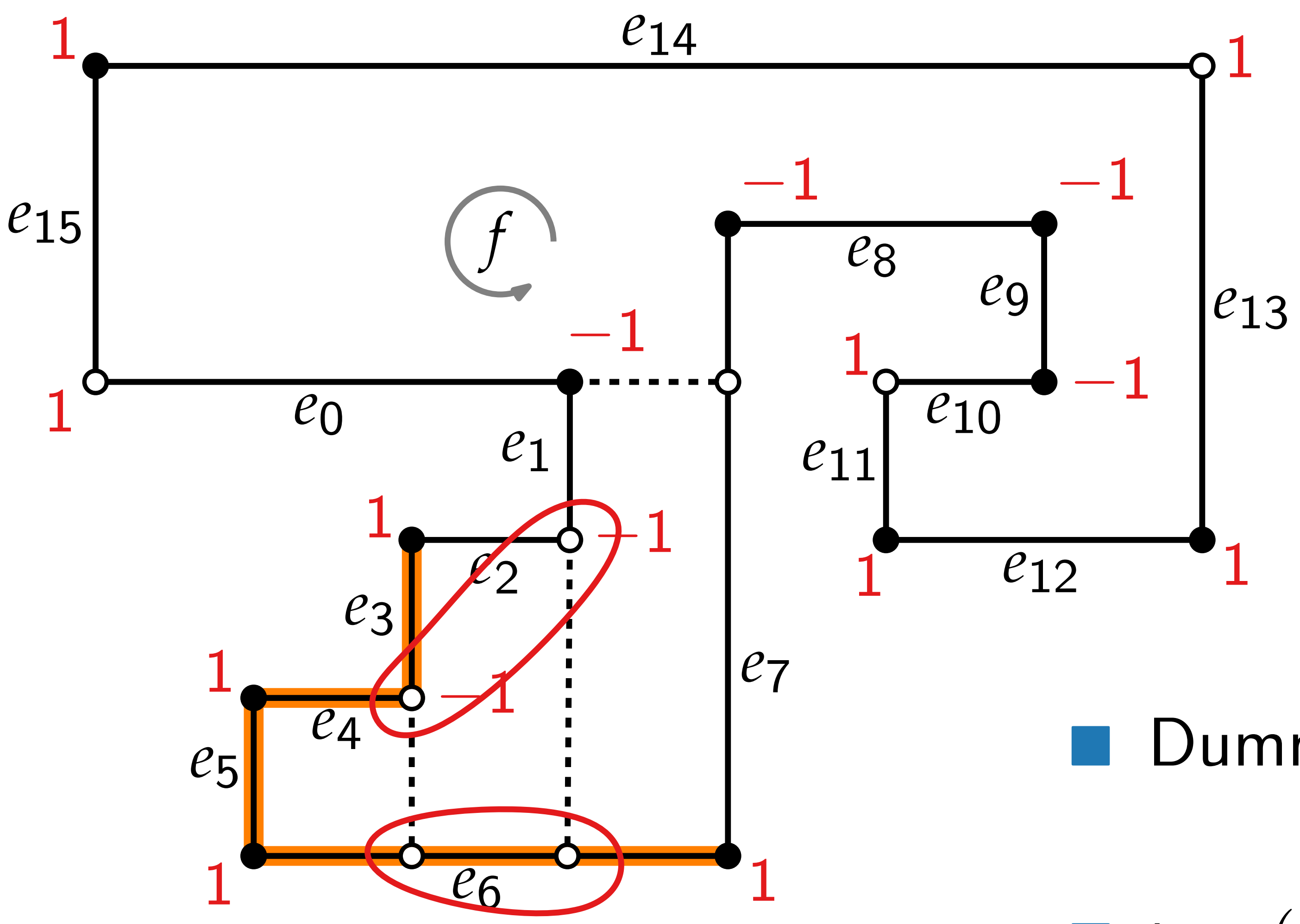

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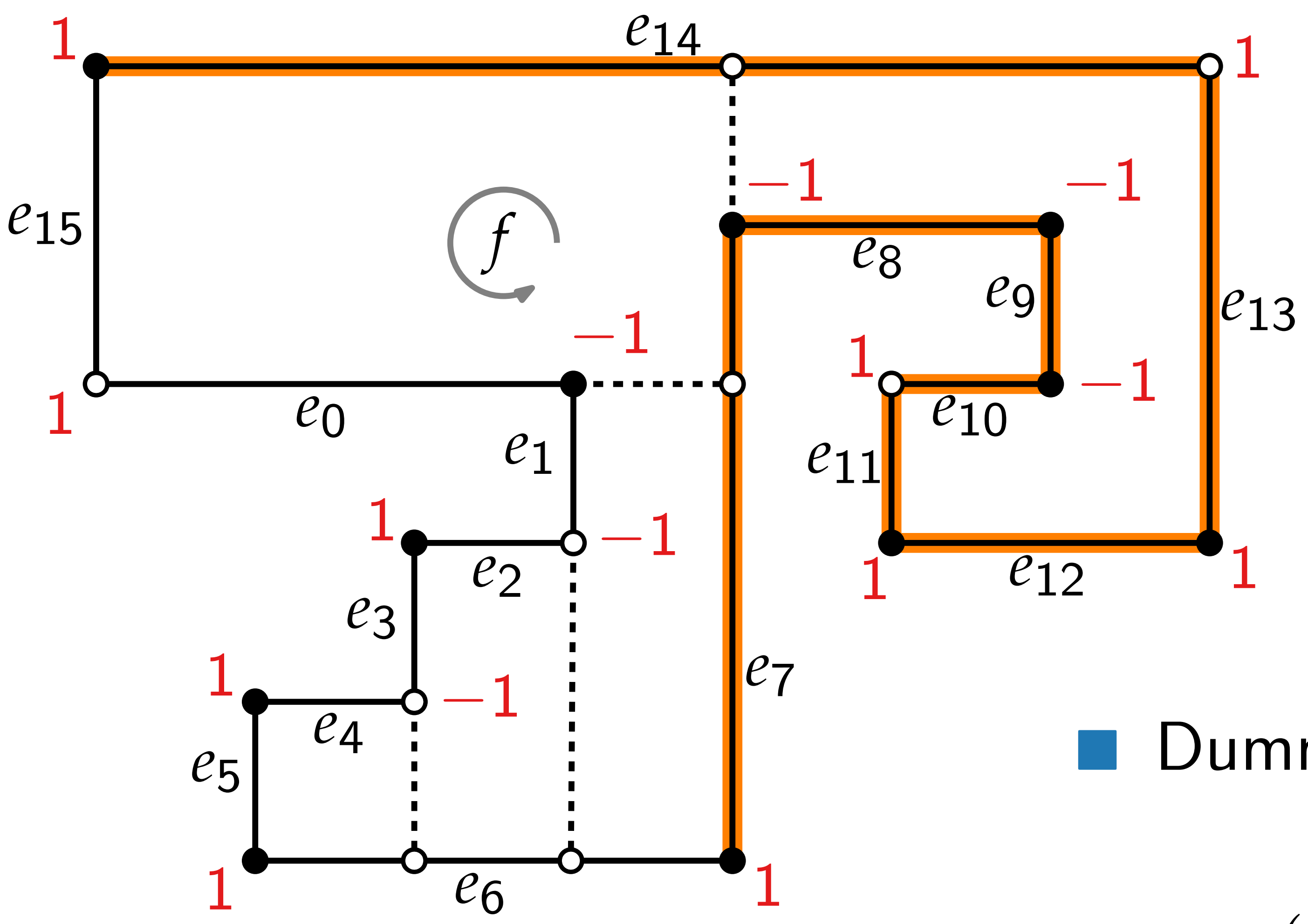

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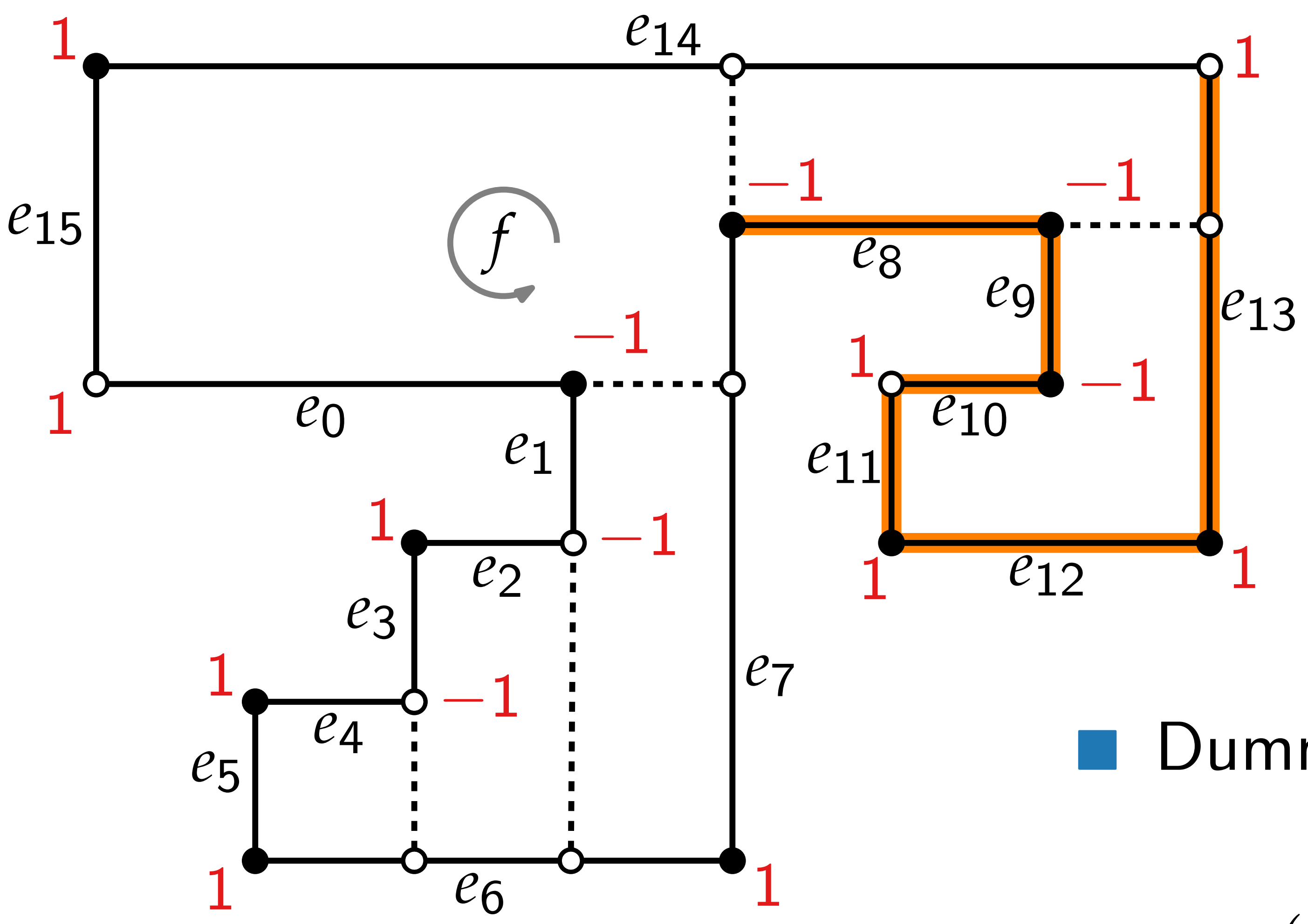

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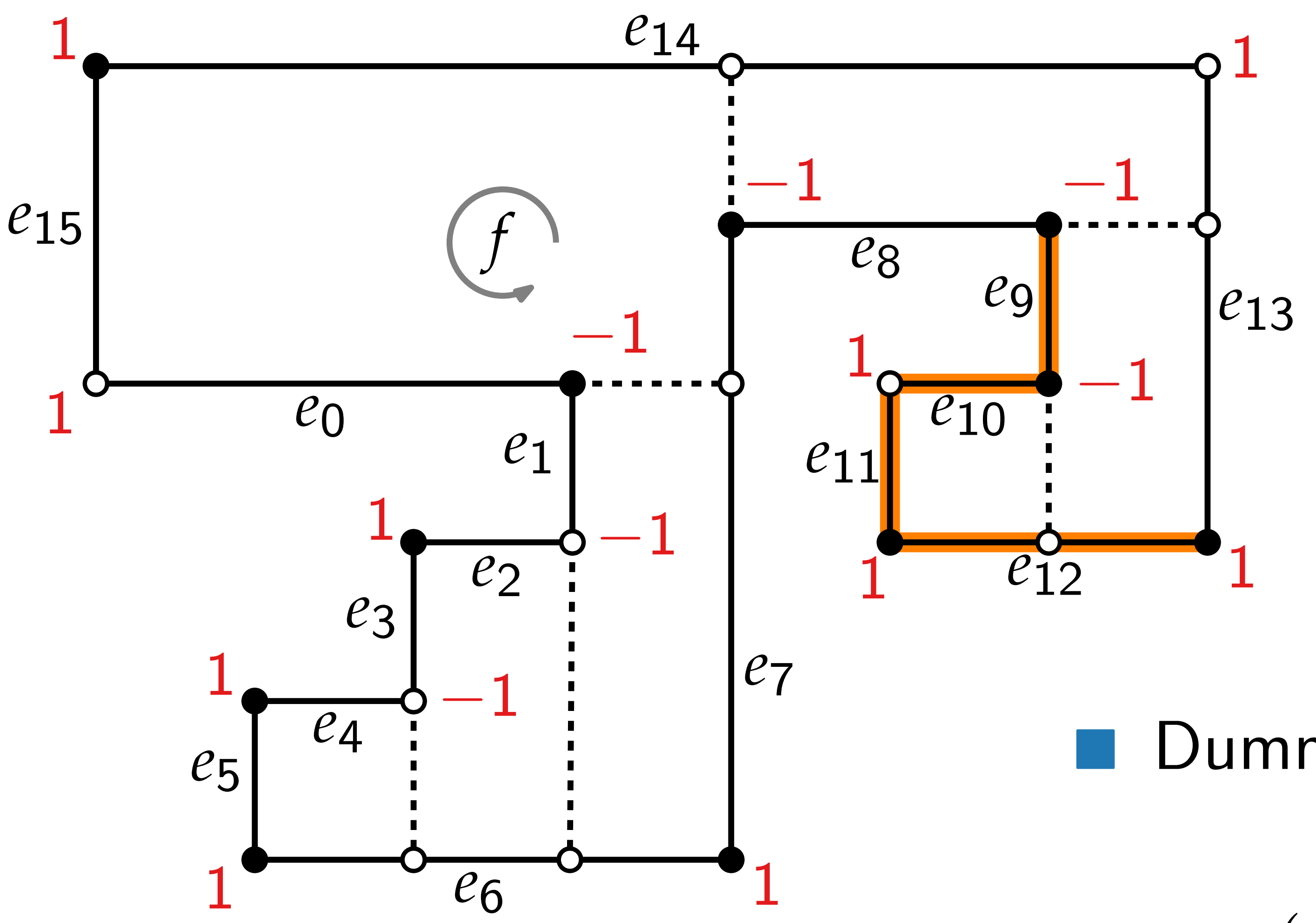

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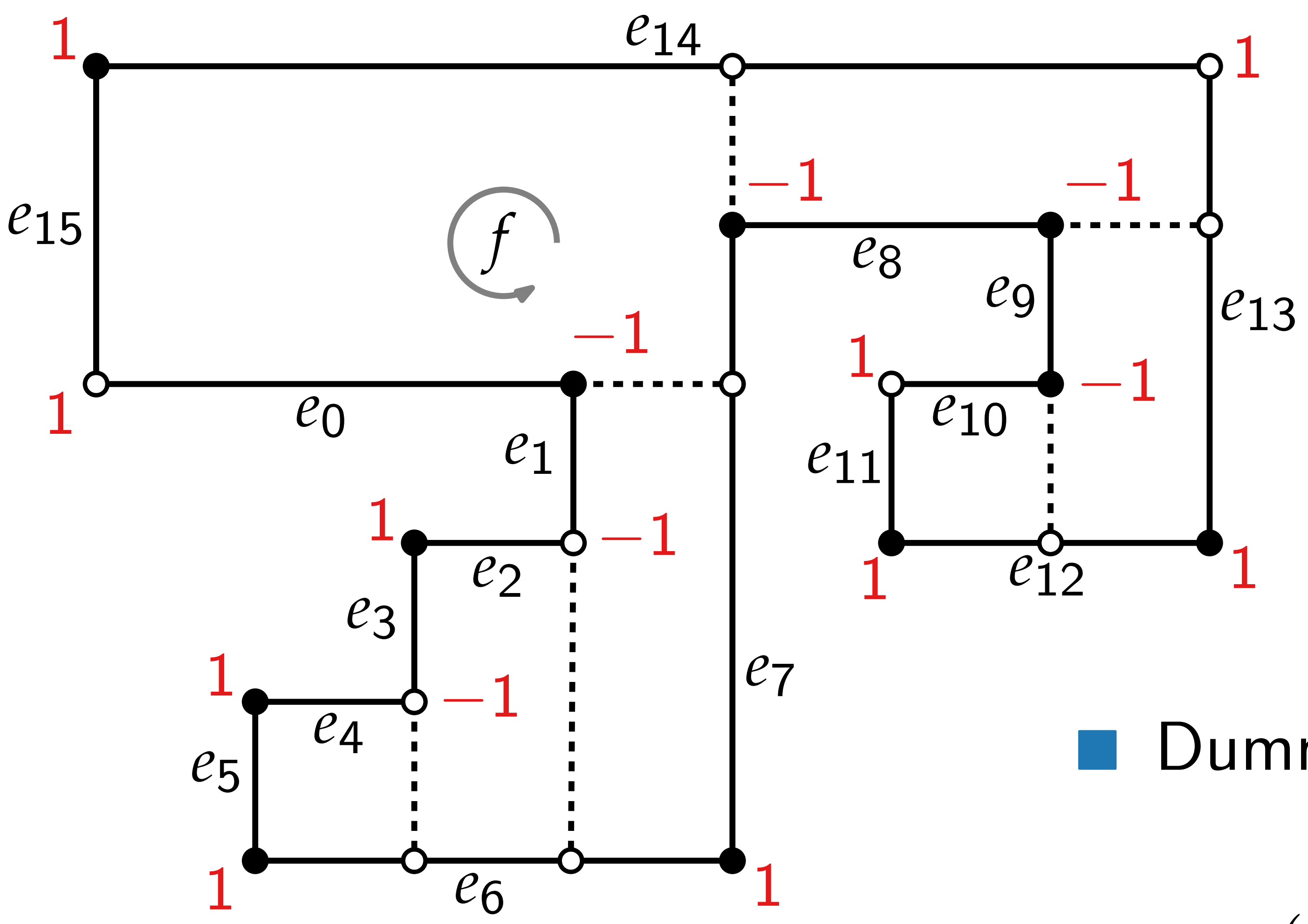

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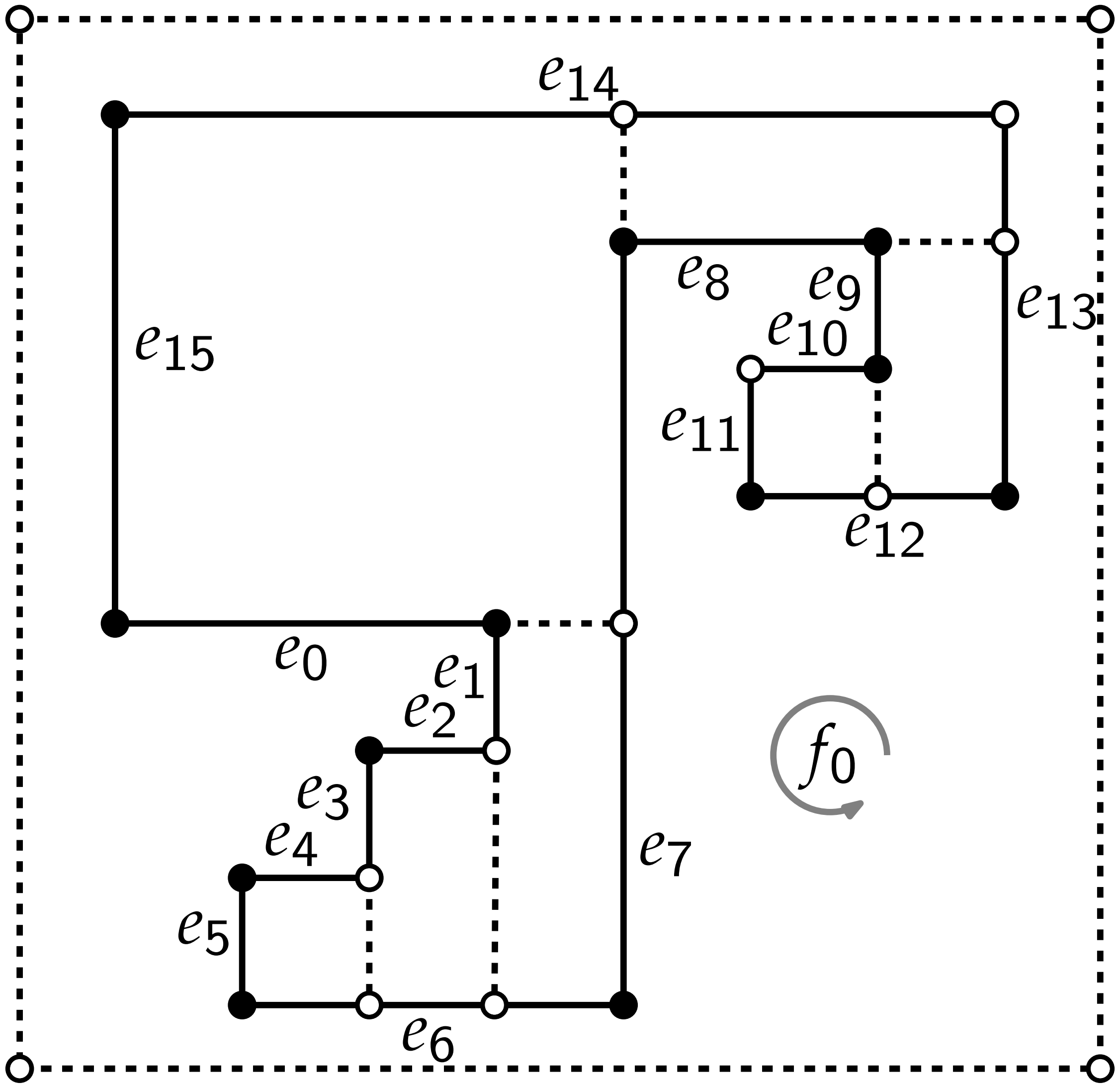

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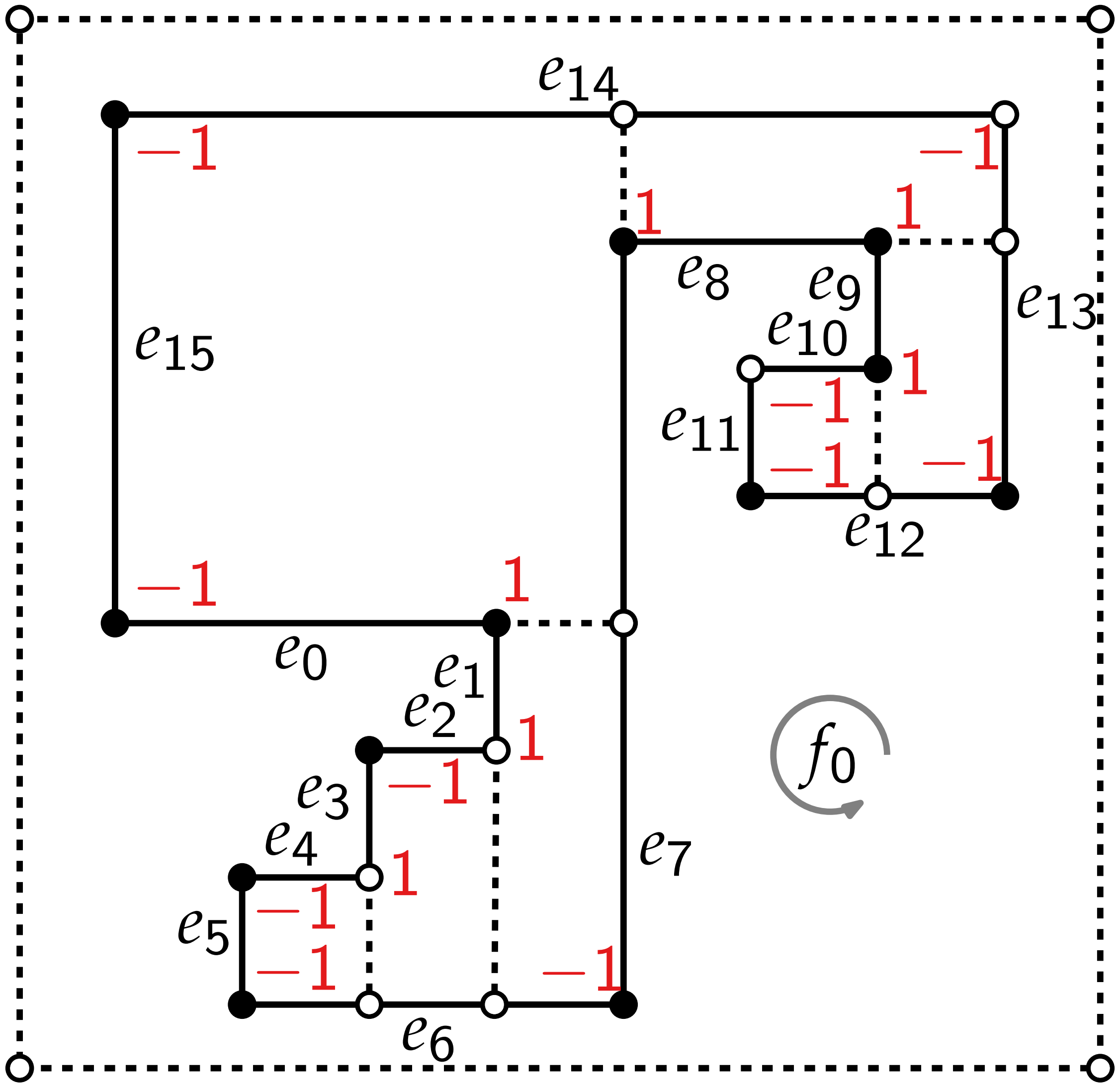

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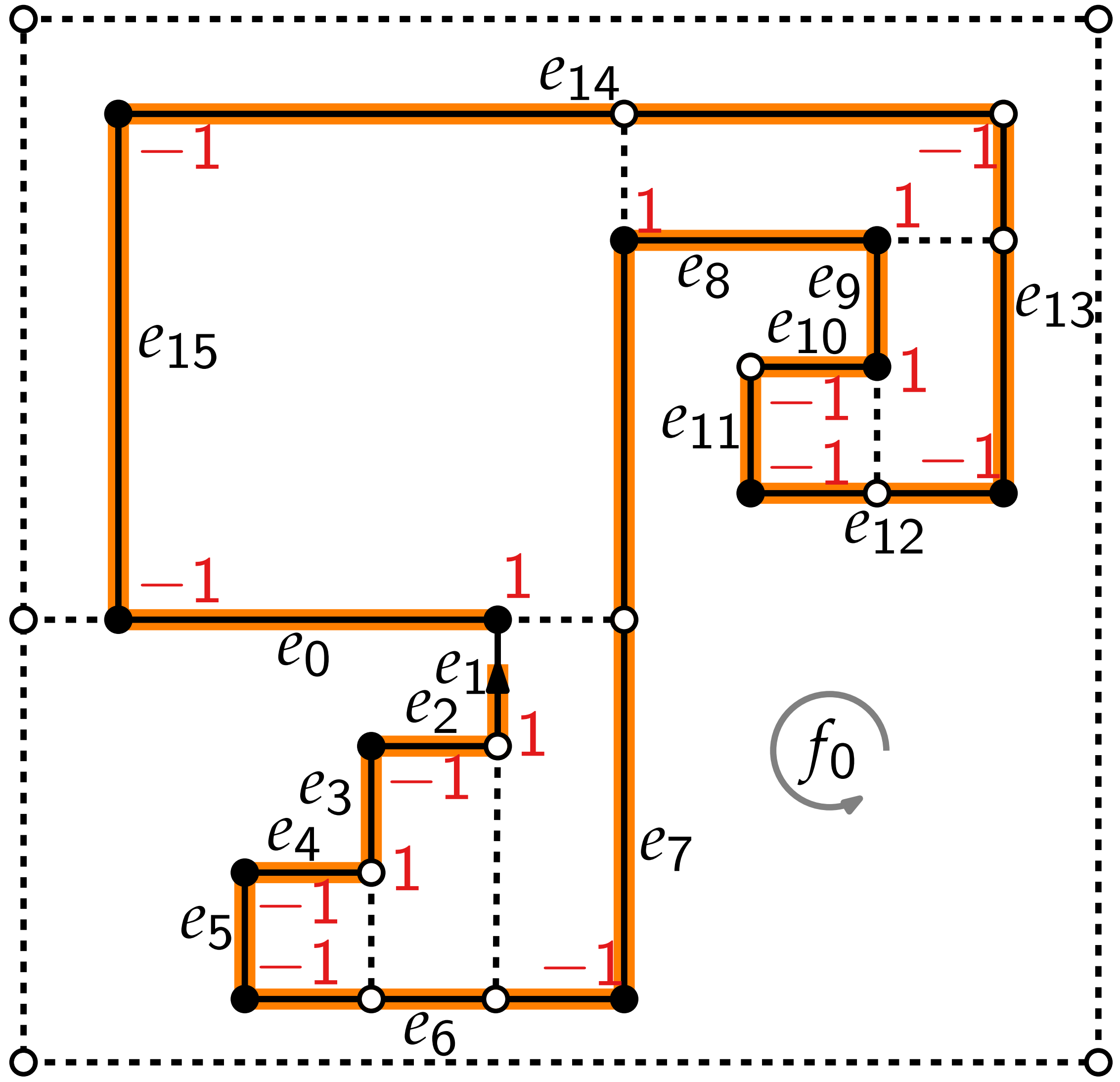

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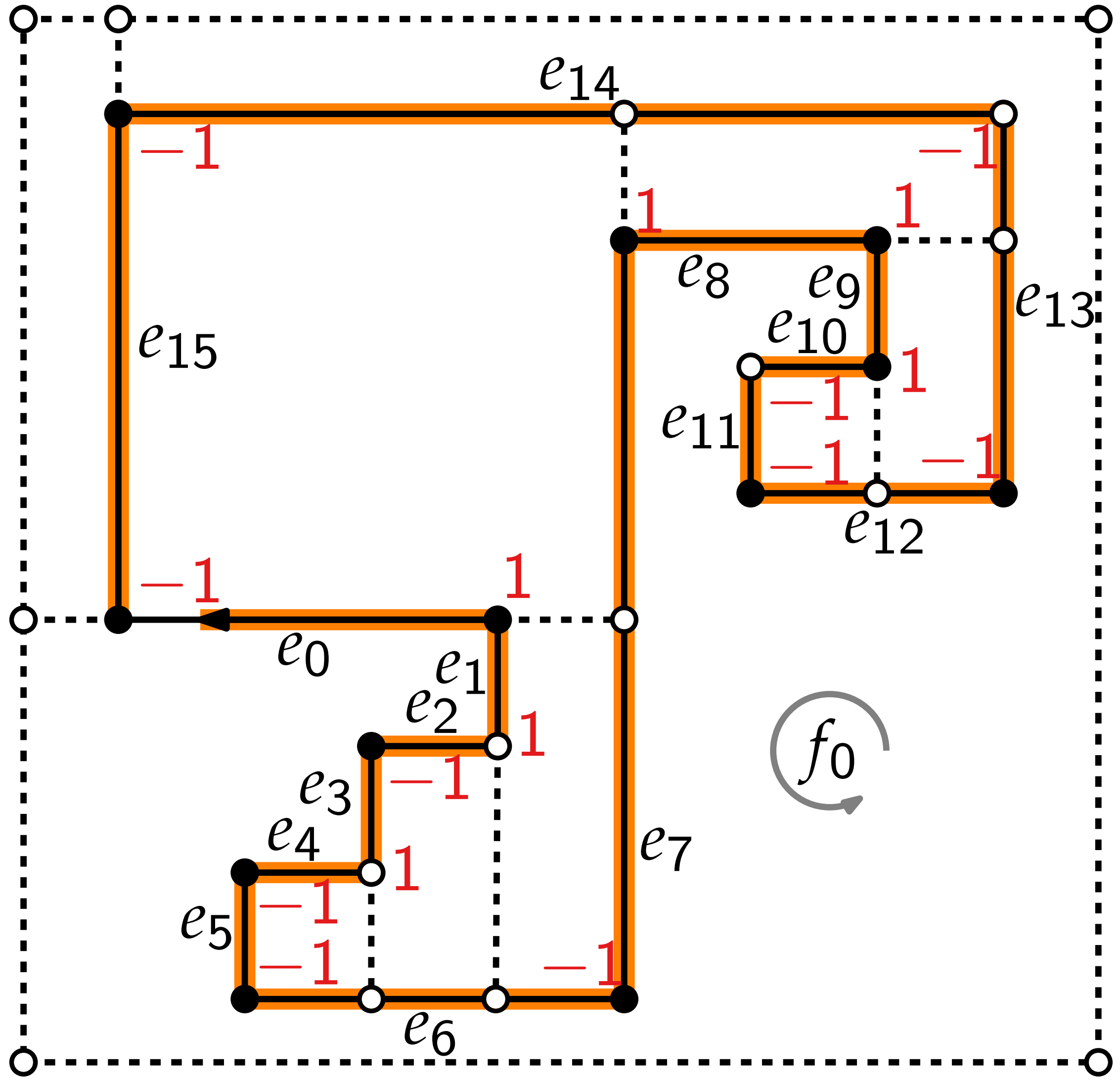

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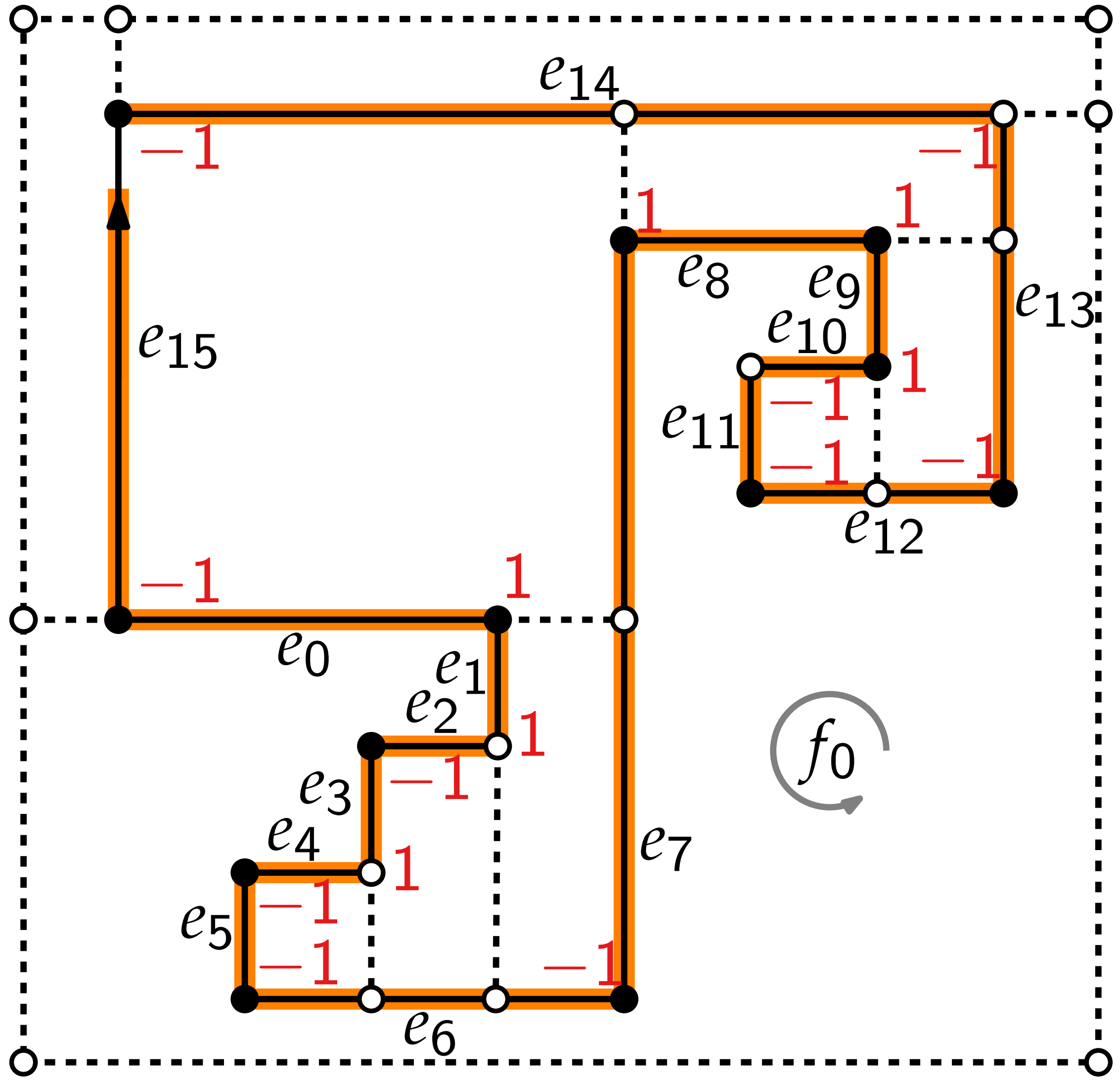

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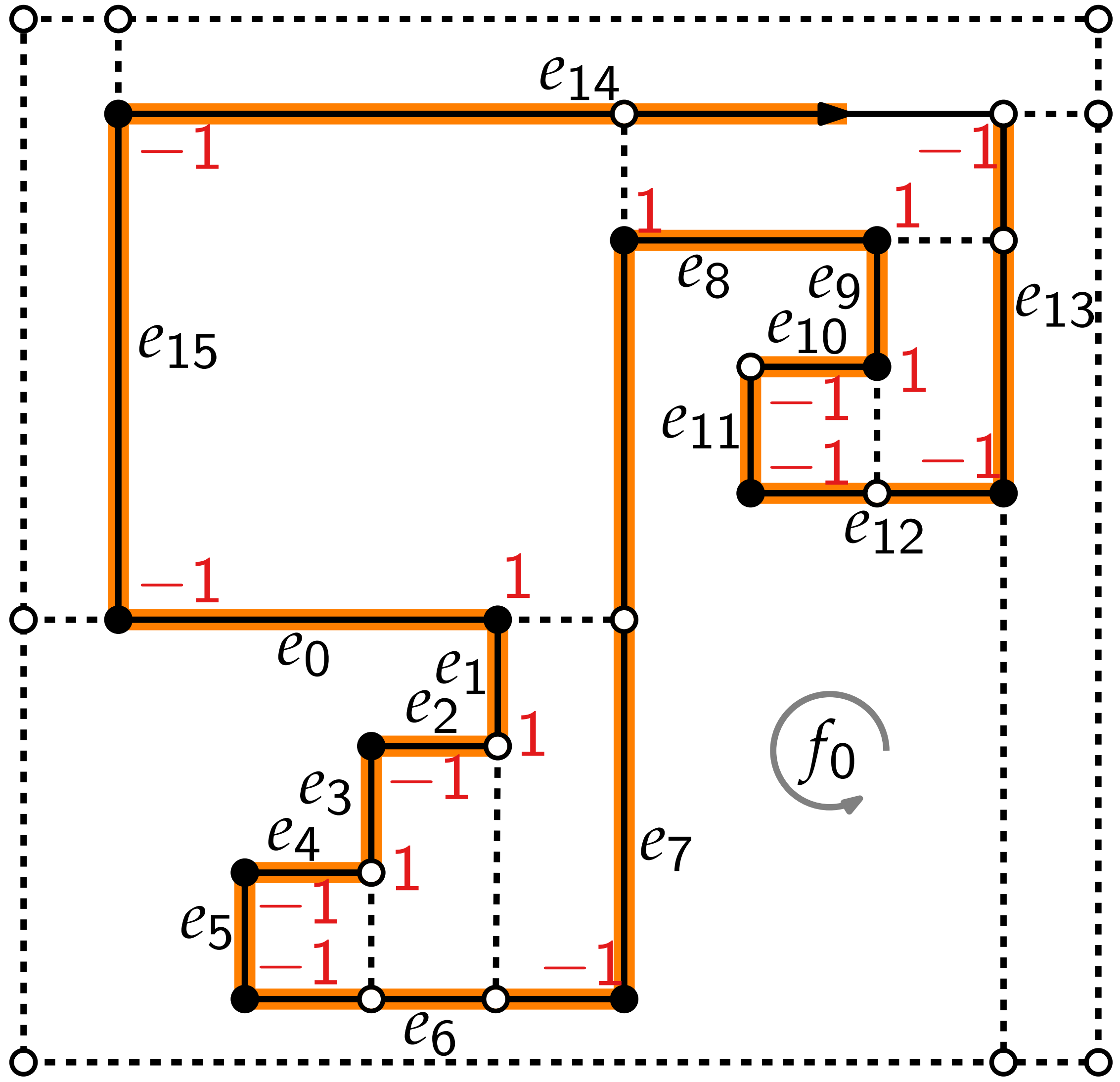

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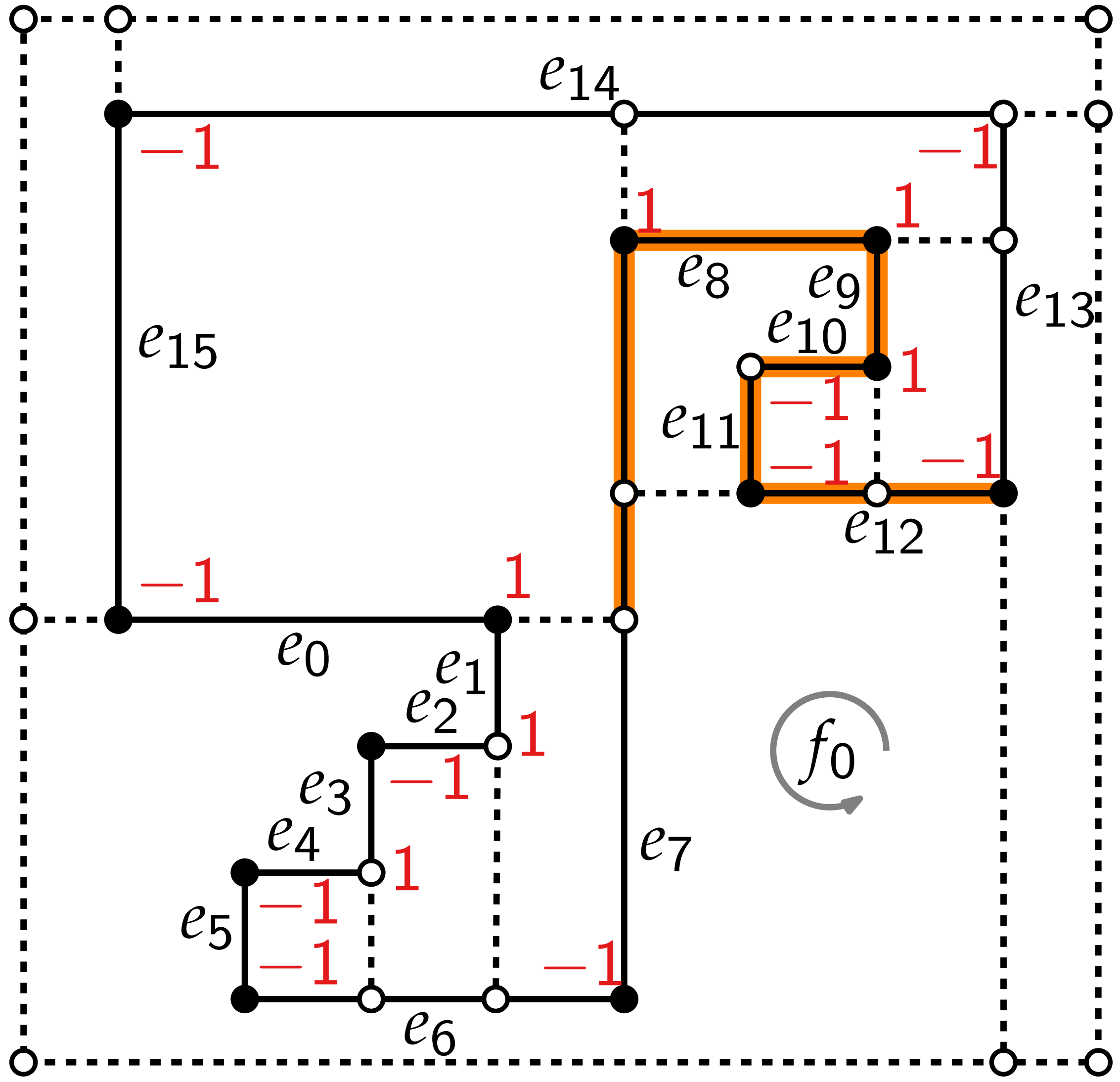

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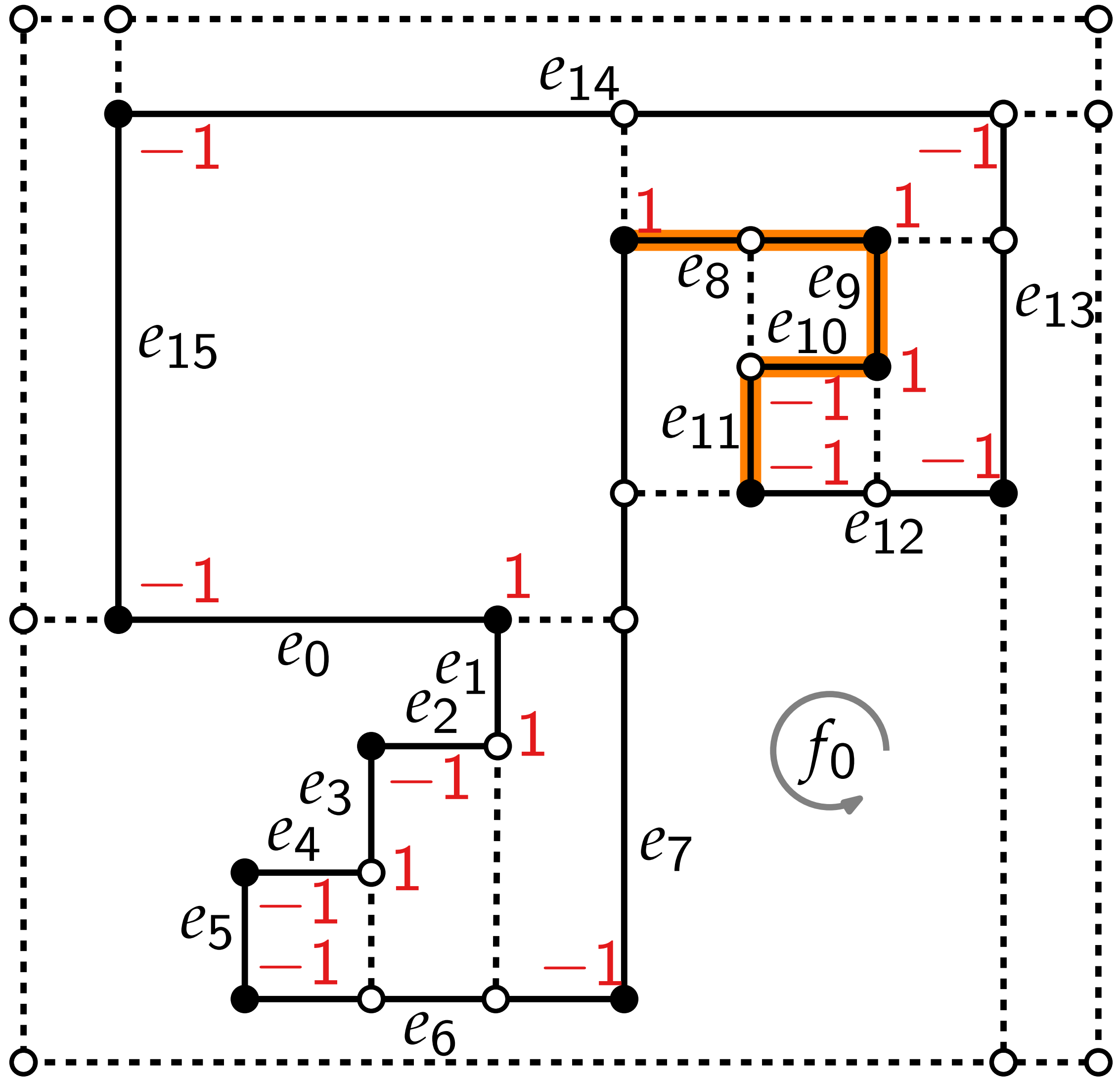

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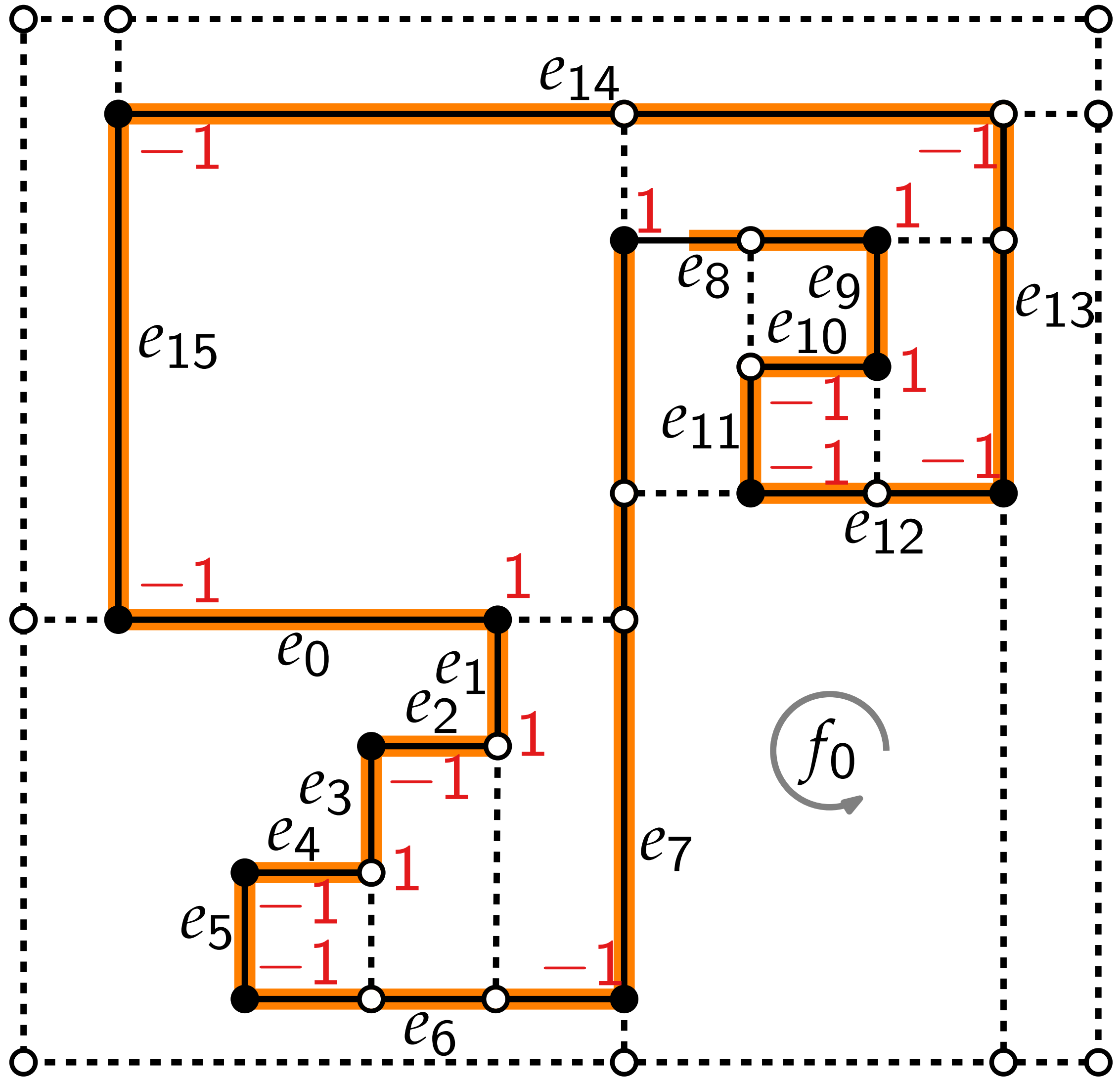

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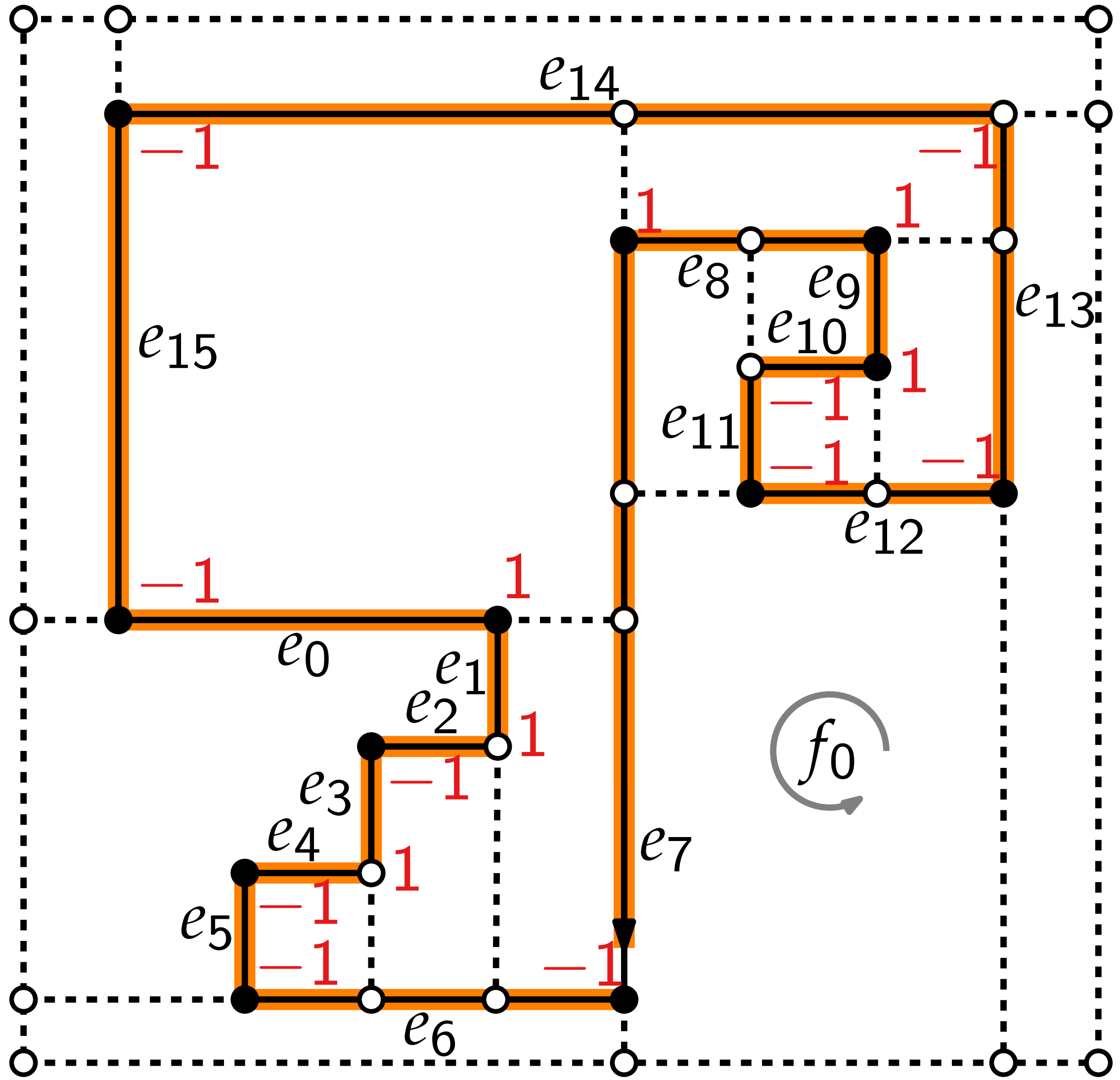

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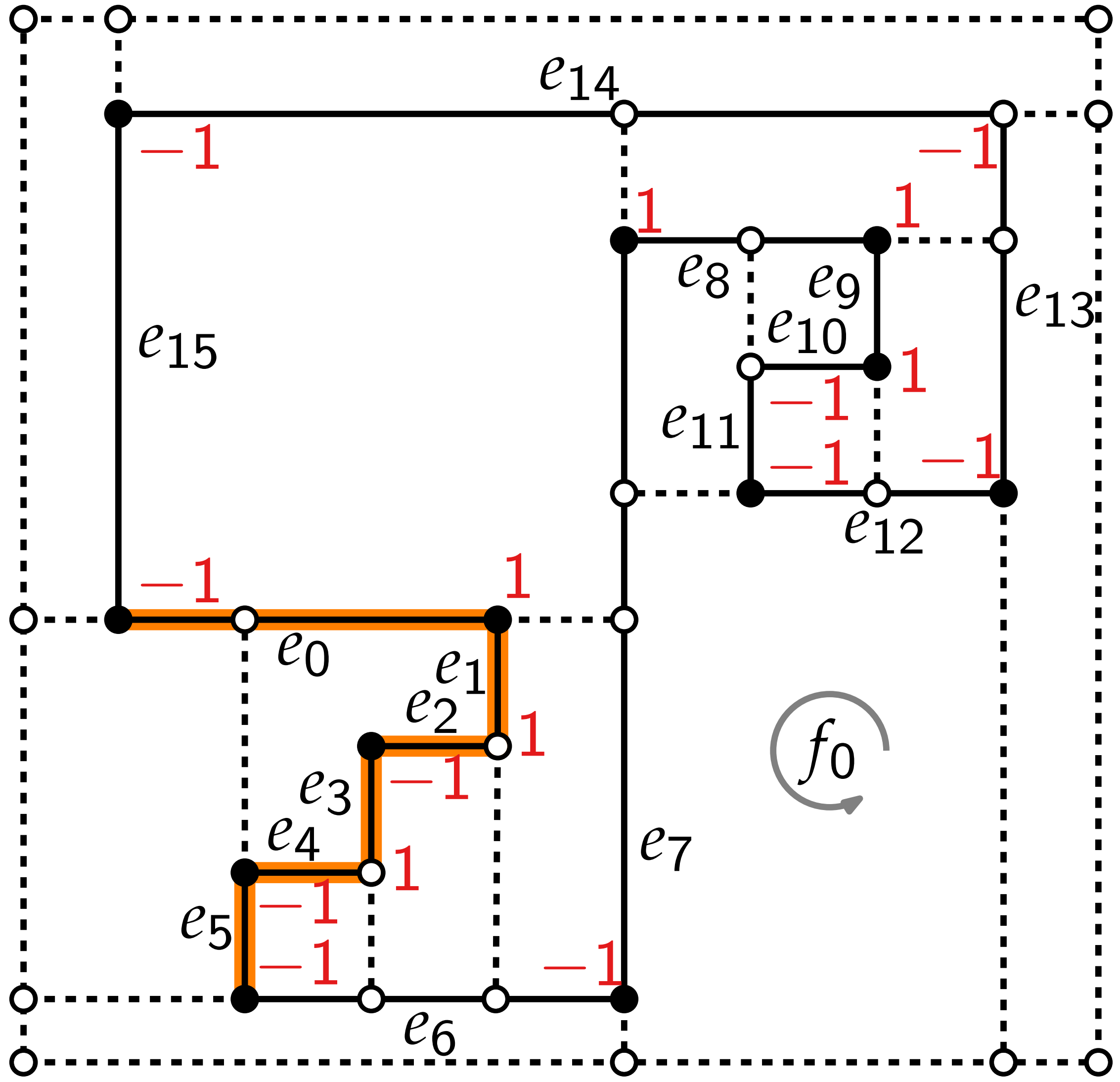

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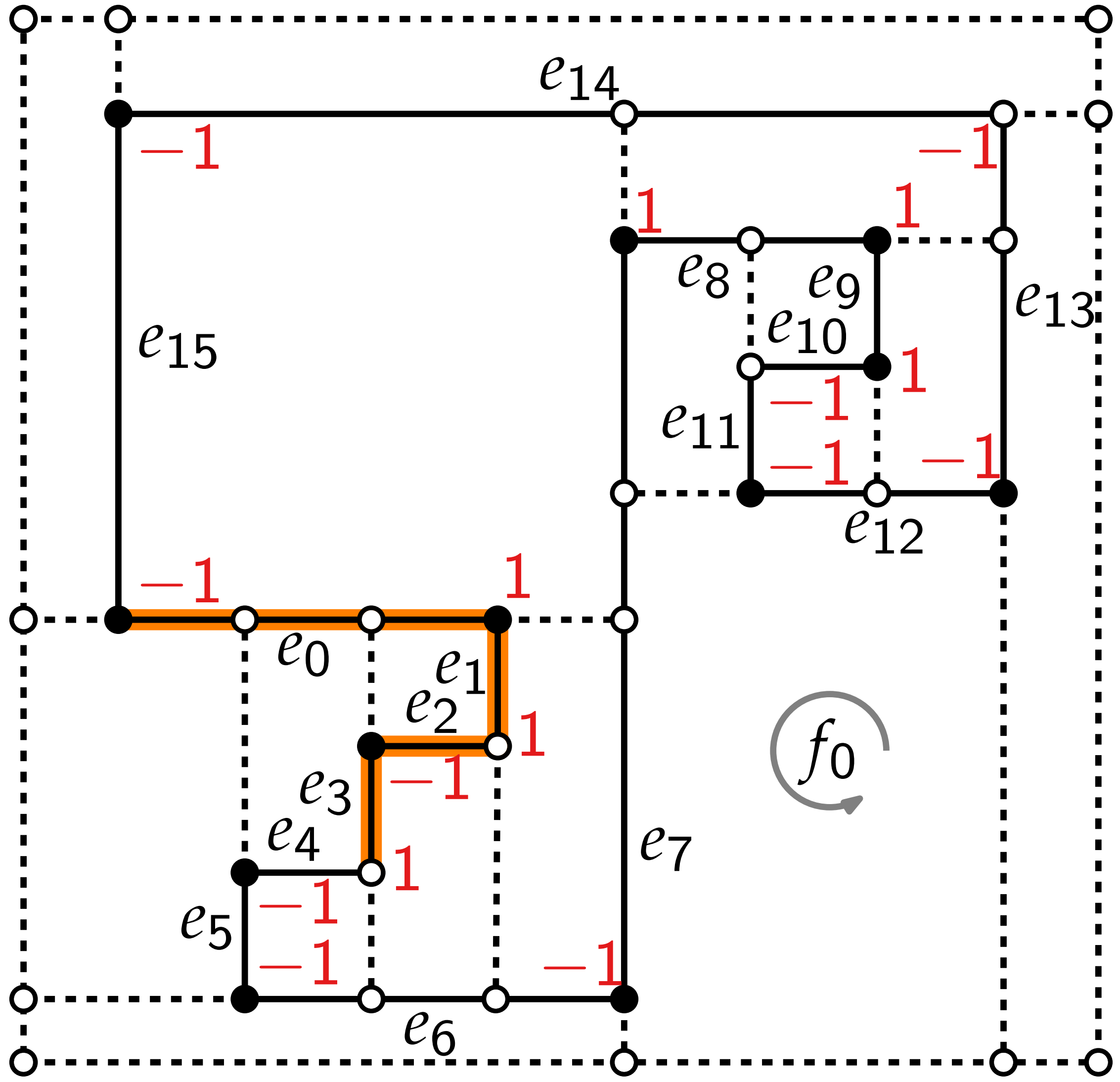

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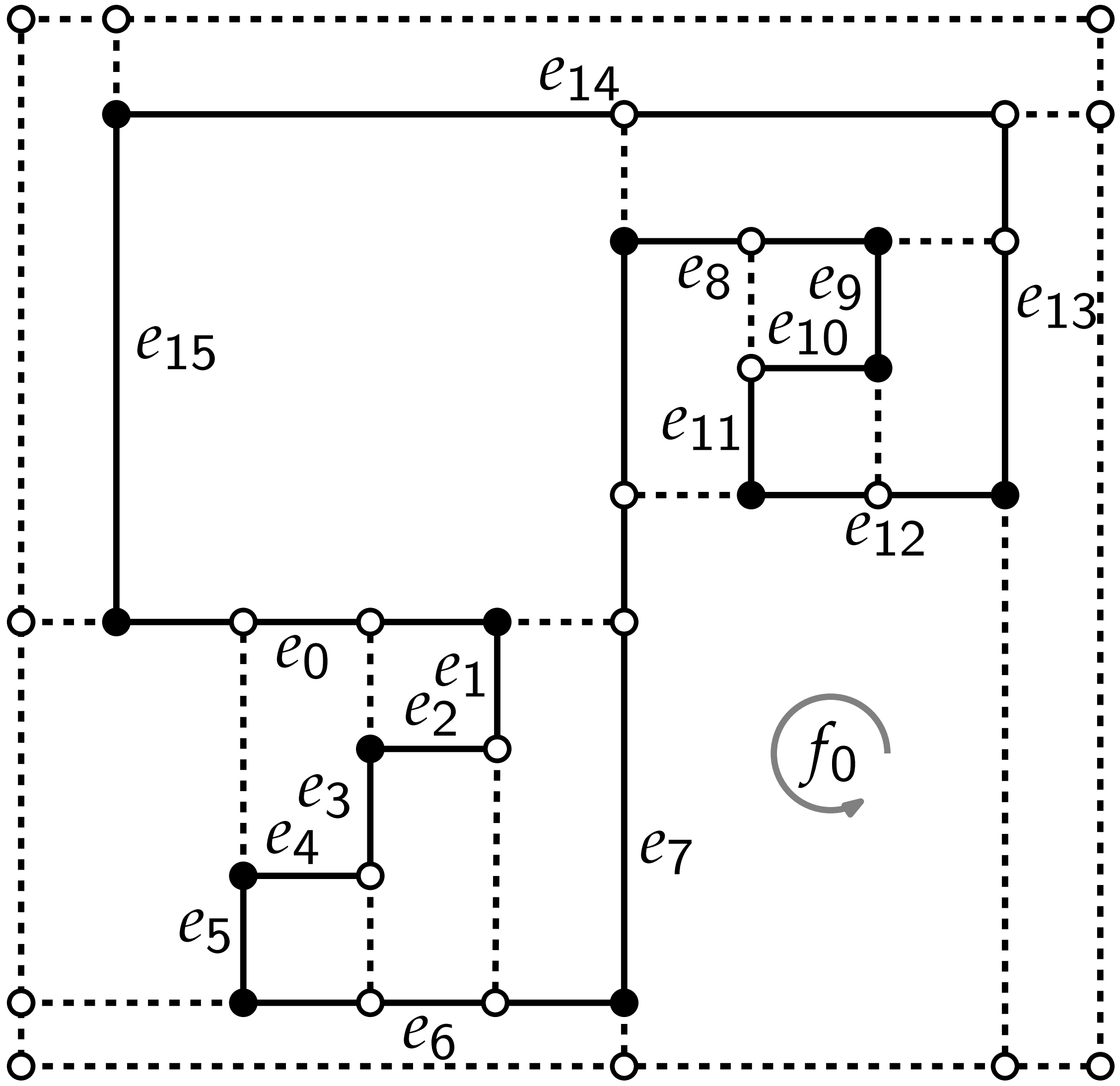

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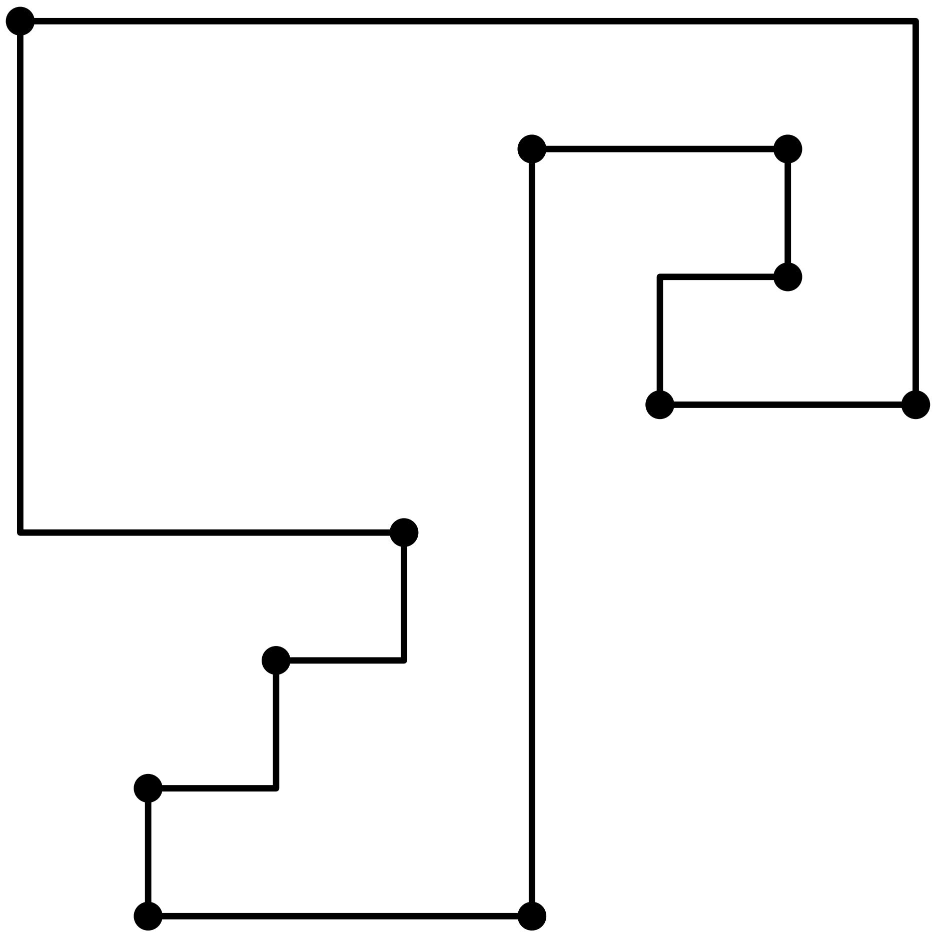

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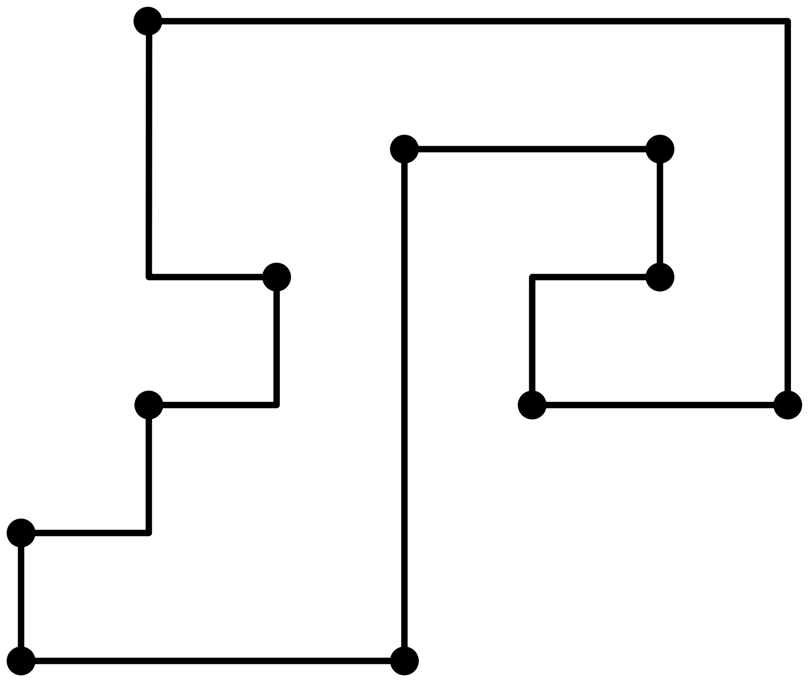

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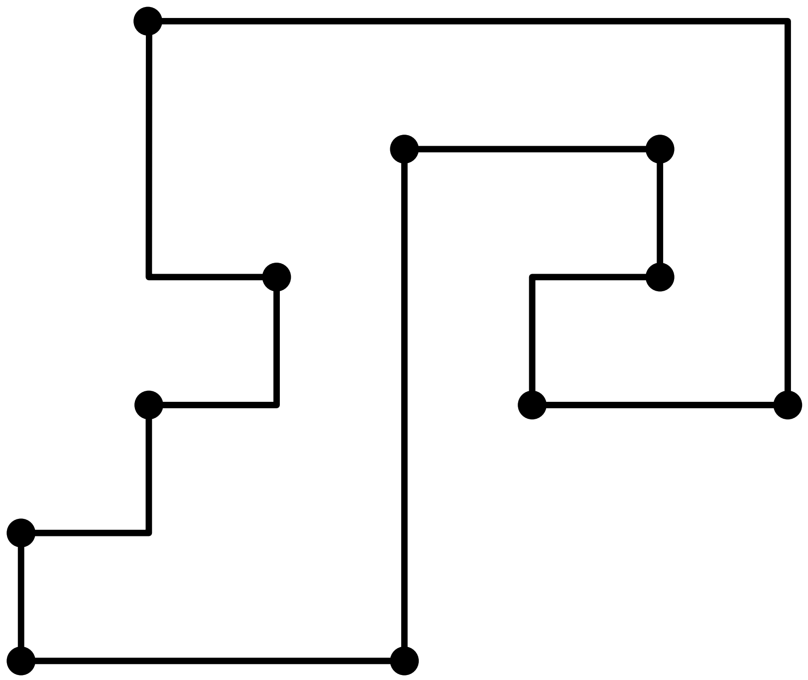

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Compact drawing?

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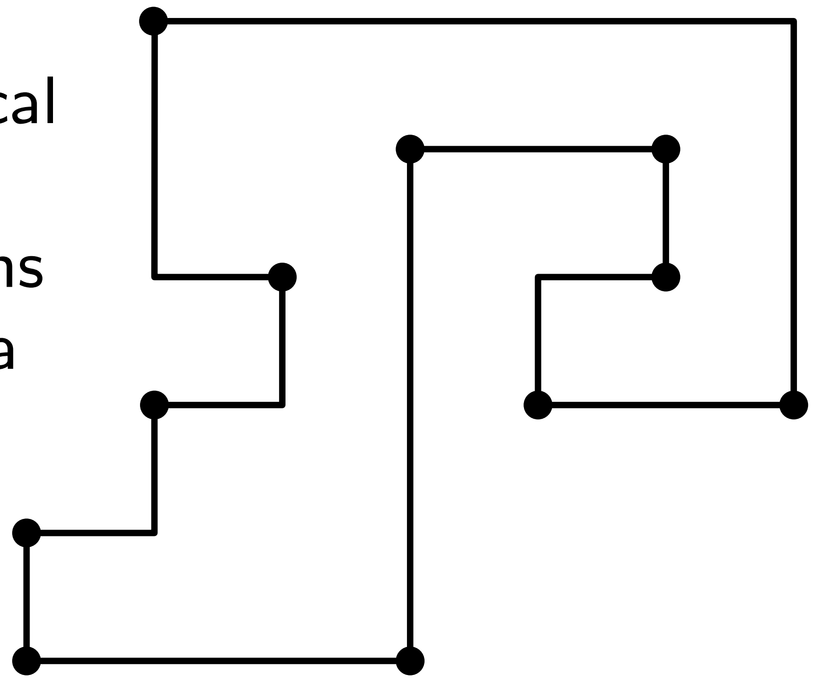
- An orthogonal drawing is **compact** if every horizontal and vertical line spanning the drawing area contains at least a vertex or a bend.



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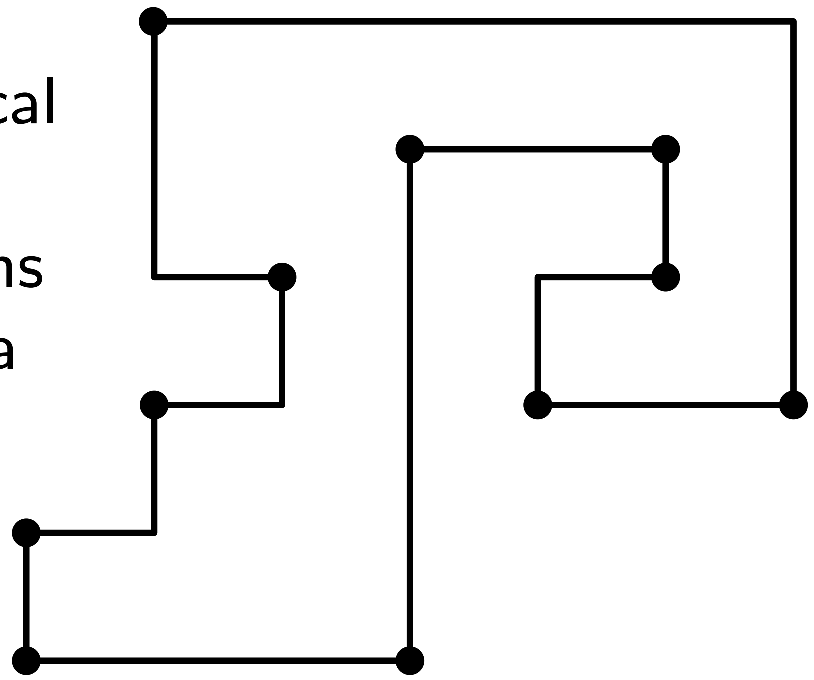

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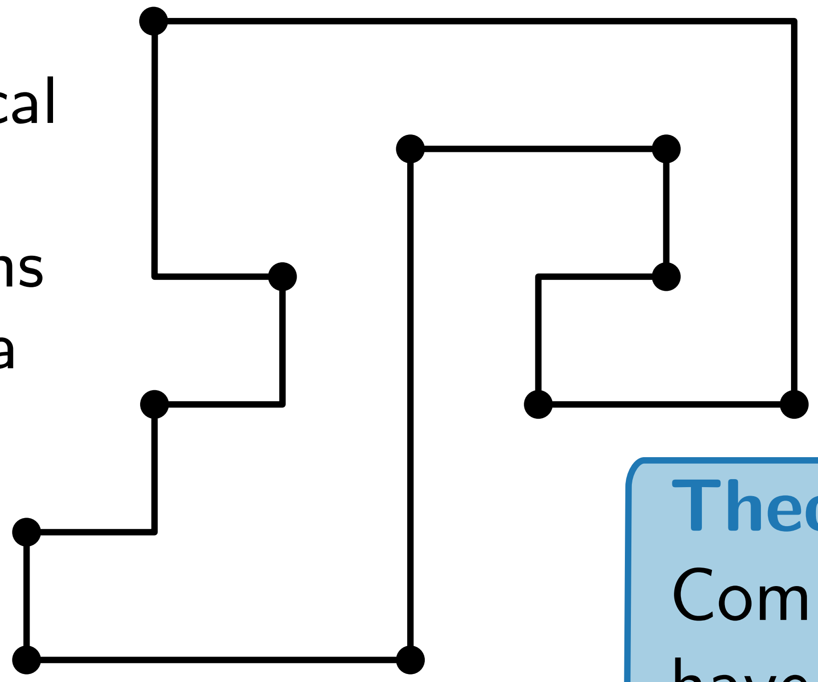

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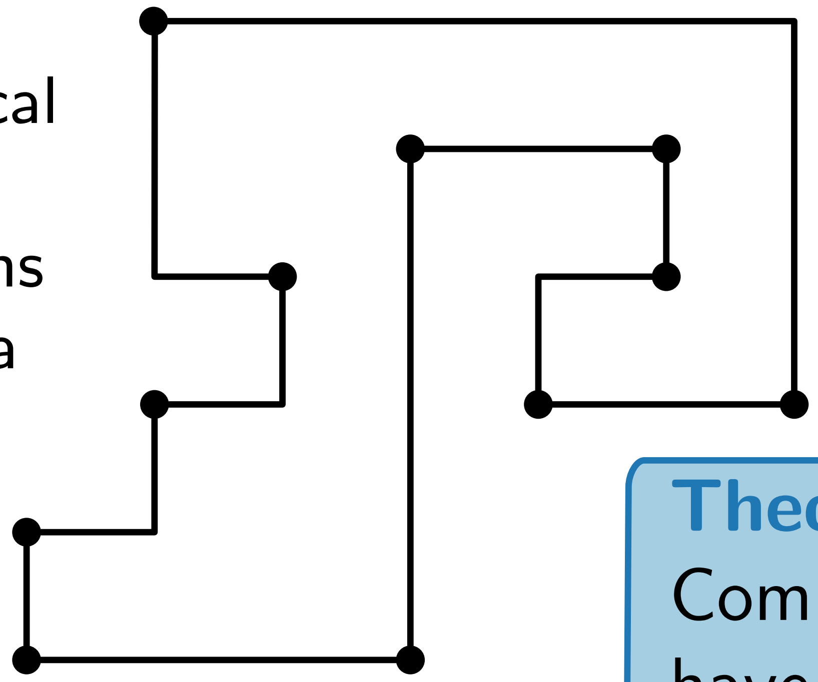

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- Compaction for given orthogonal representation is in general NP-hard.

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Let D be a compact orthogonal plane drawing of a graph G with b bends. Let W and H be the width and hight of the grid, resp. Then, $W + H \leq b + 2n - m - 2$

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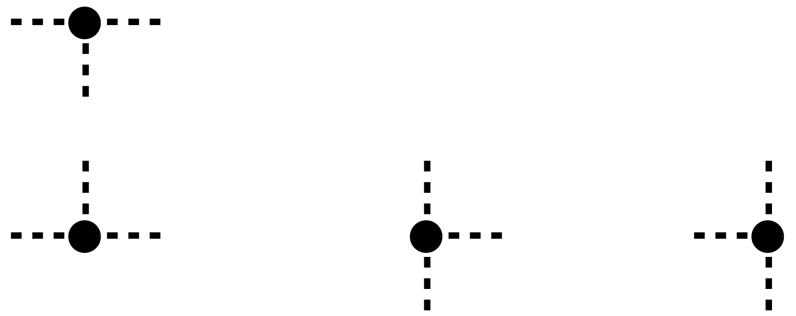

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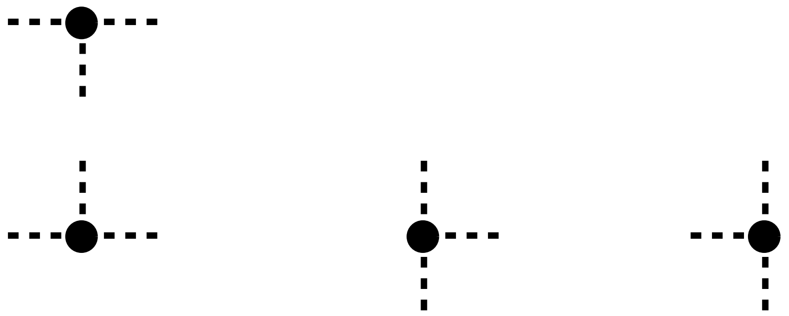

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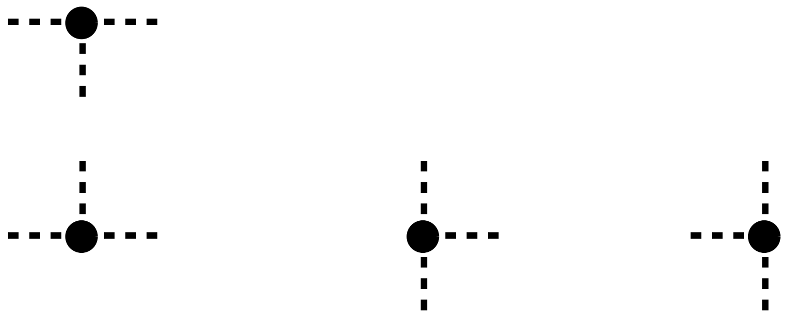

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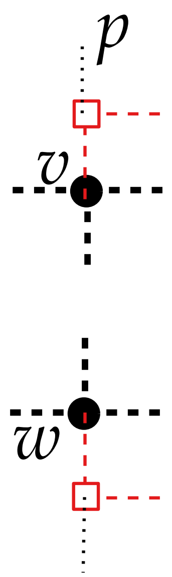

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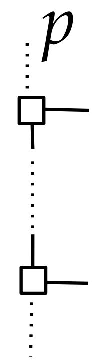

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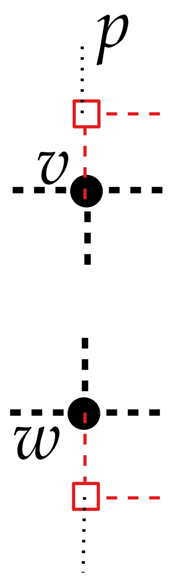

Let v, w be the top-most and bottom most ones, resp.

– If v is not using its top port, it contributes "1" unit to $Top(D)$.

– if v uses its top port, it contributes "1" unit to b.

$\Rightarrow$ Vertex v contributes at least "1" unit to $b + Top(D)$.

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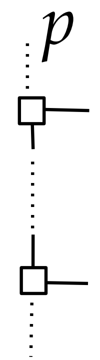

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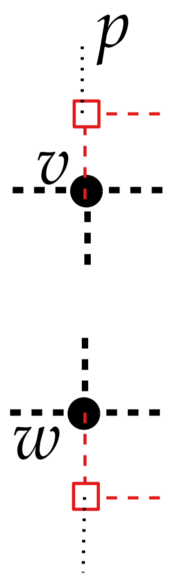

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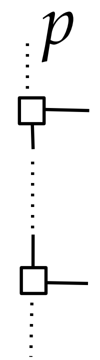

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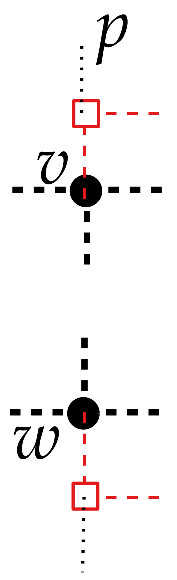

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We conclude:

Column p contributes at least “2” units to $b + Top(D) + Bottom(D)$.

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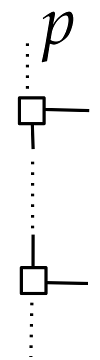

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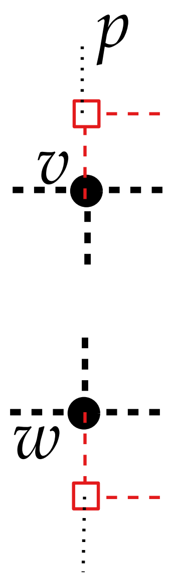

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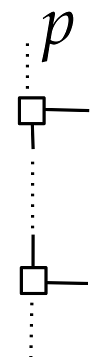

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Thus,

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  - n variables $x_1, \ldots, x_n$
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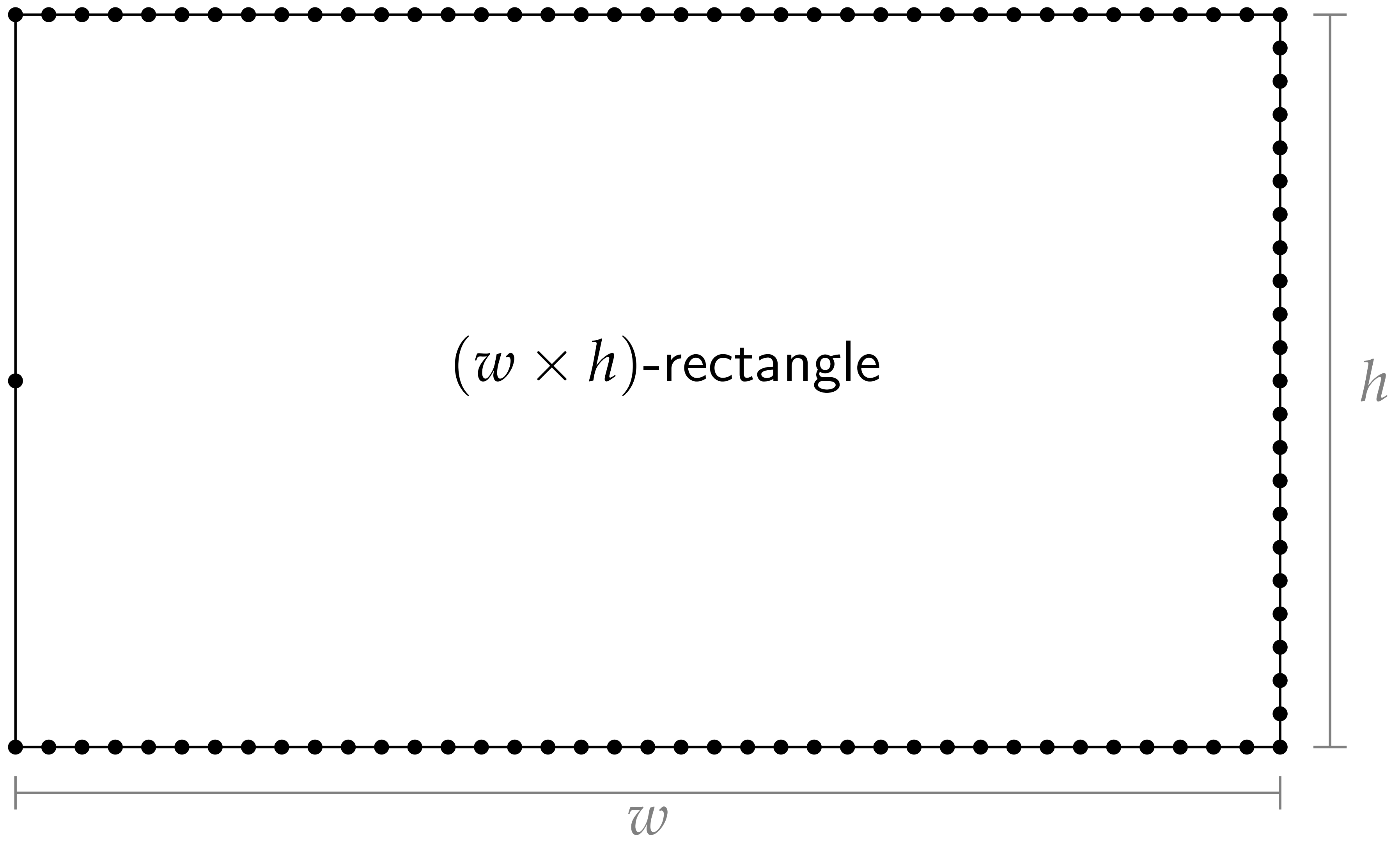
- High level structure of (G, H)
  - boundary
  - belts, and pistons
  - clause gadgets
  - variable gadgets

Boundary, belt, and "piston" gadget



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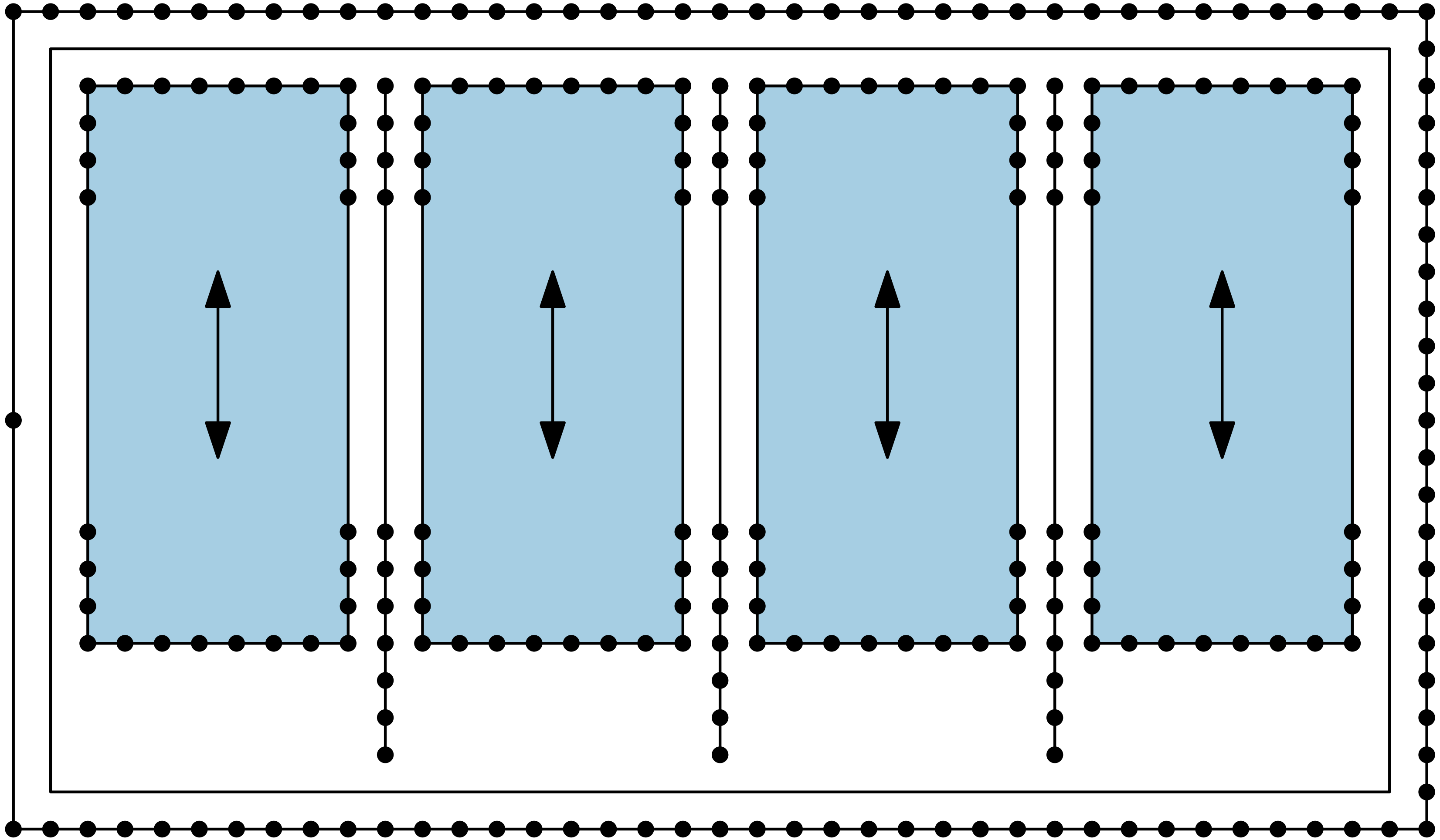

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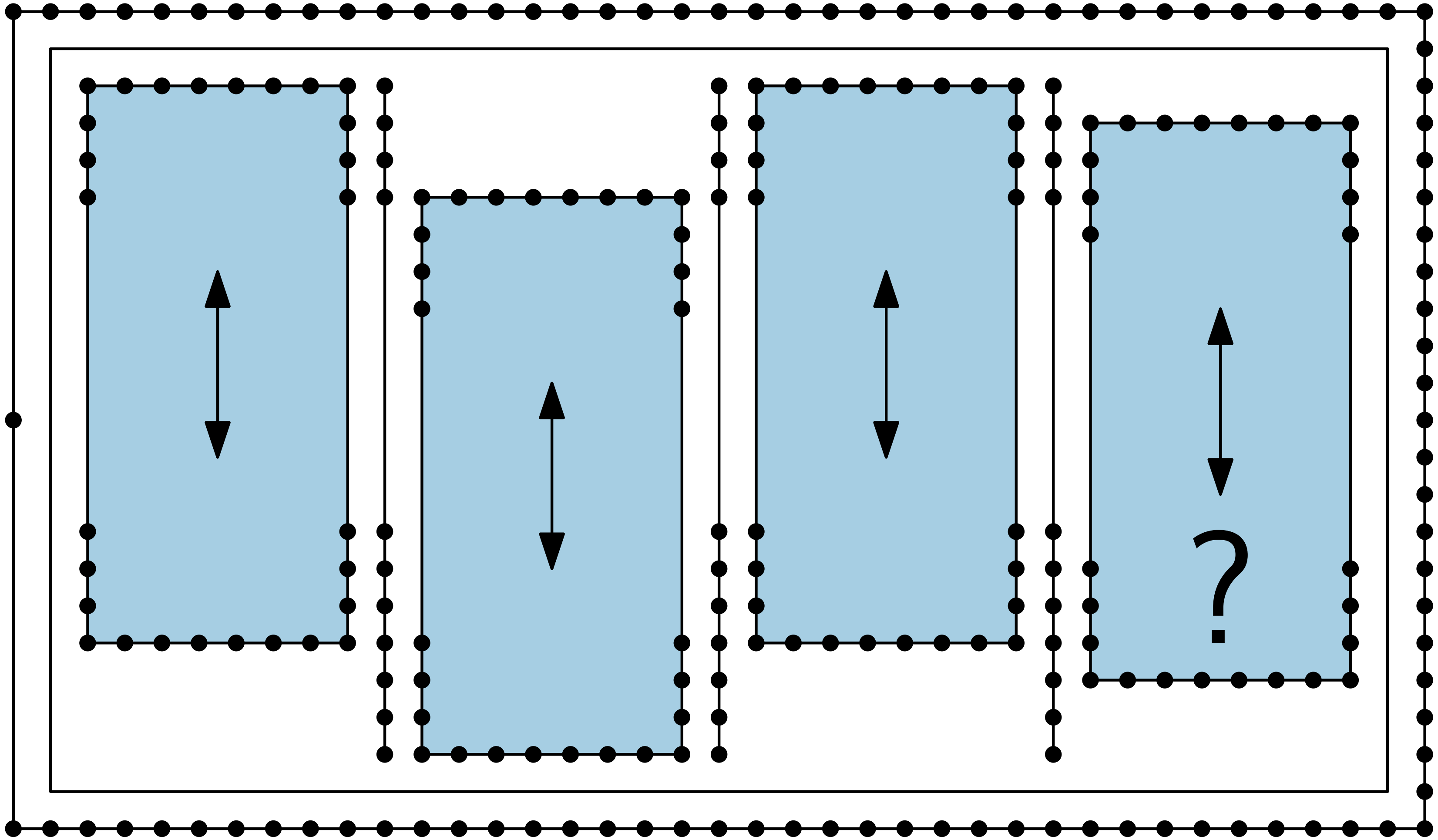

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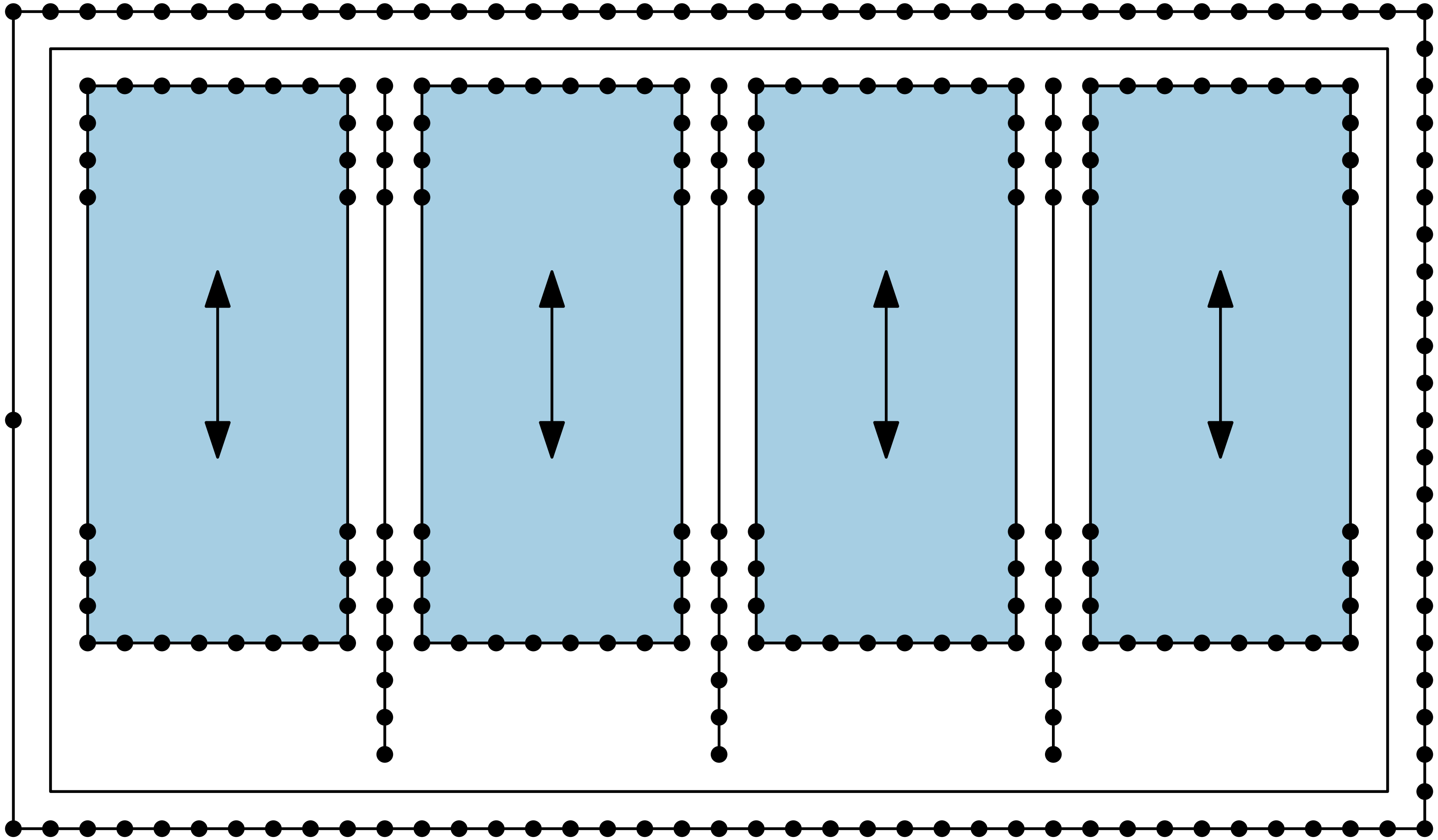

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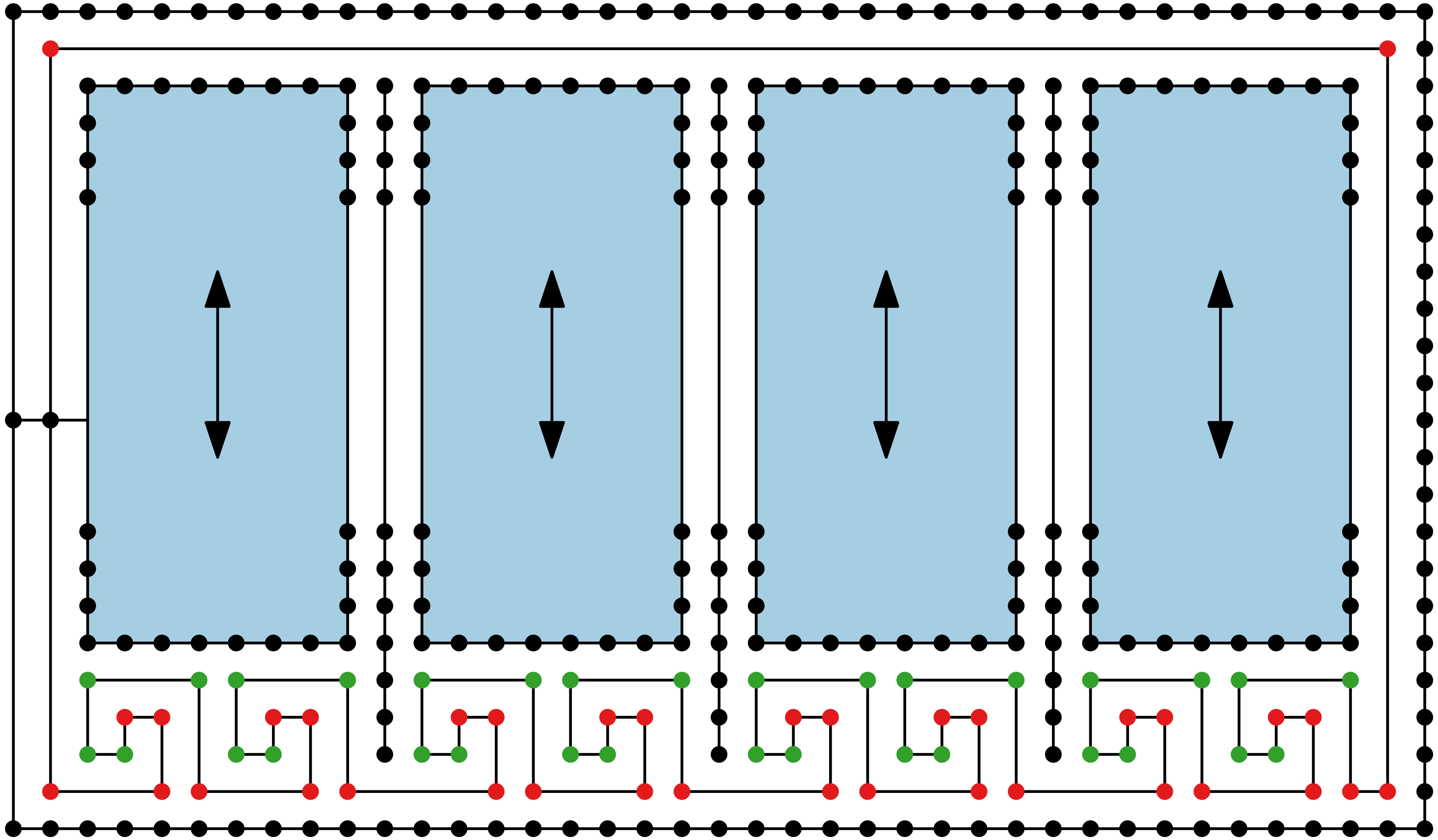

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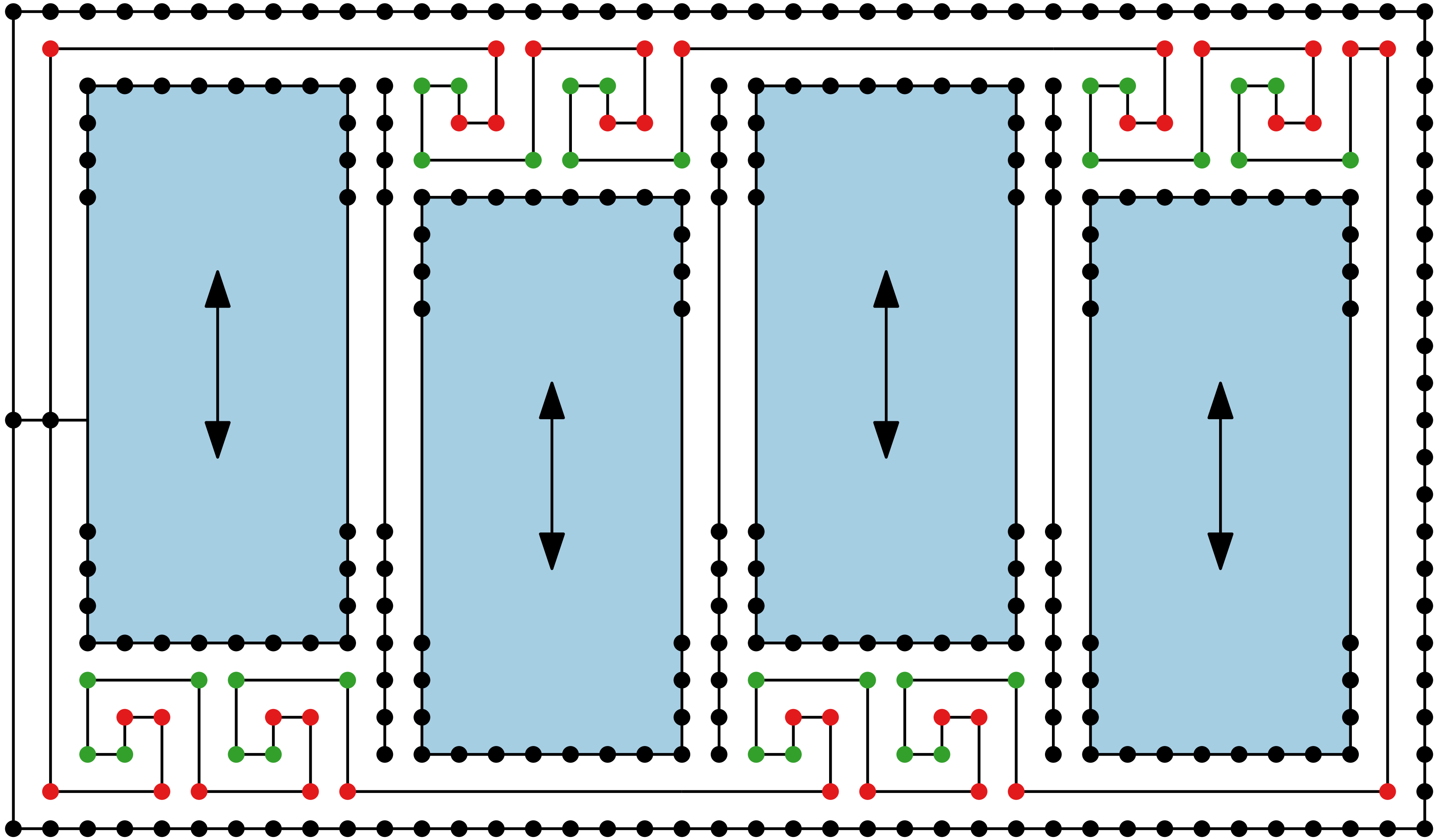

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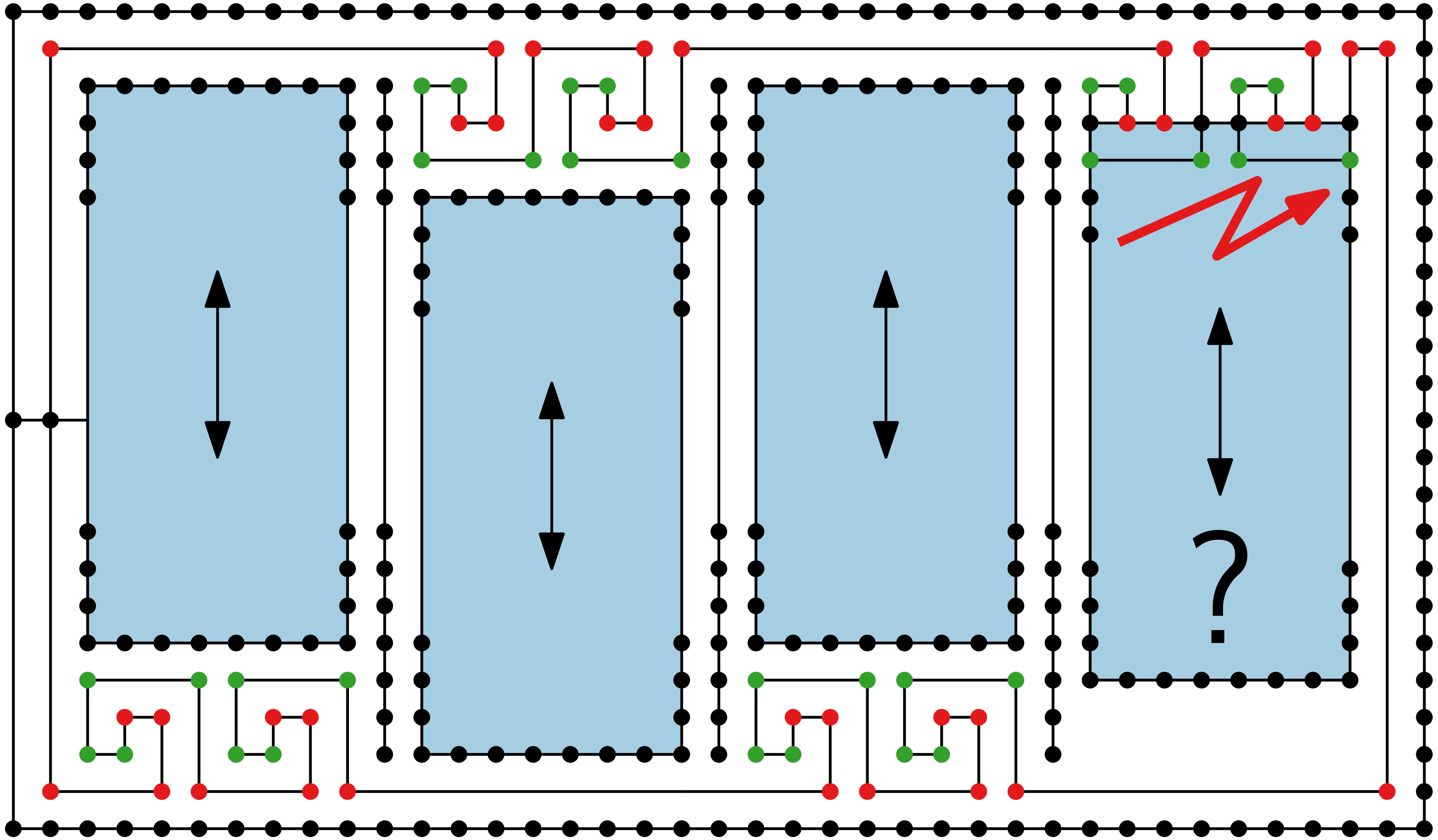

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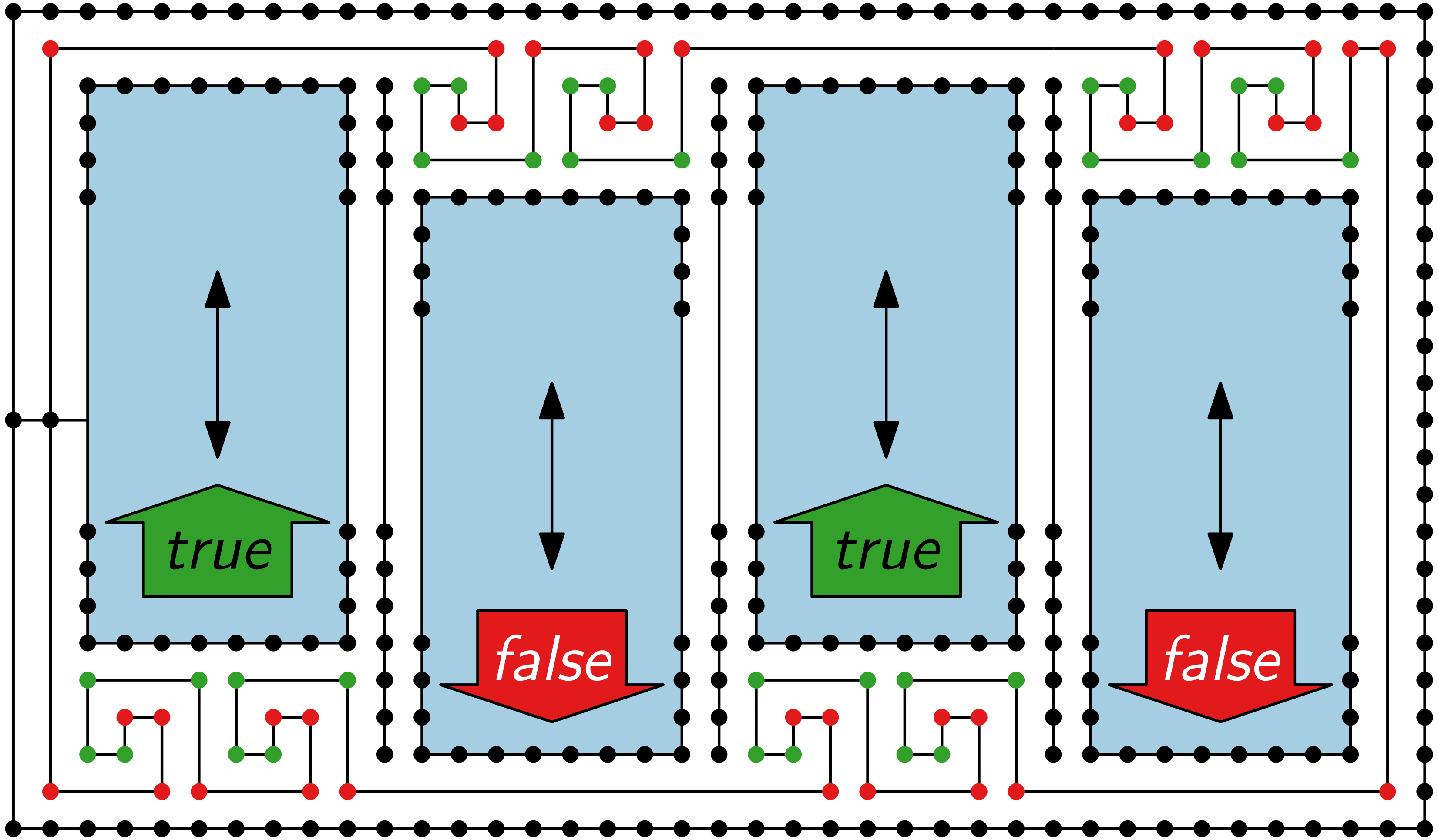
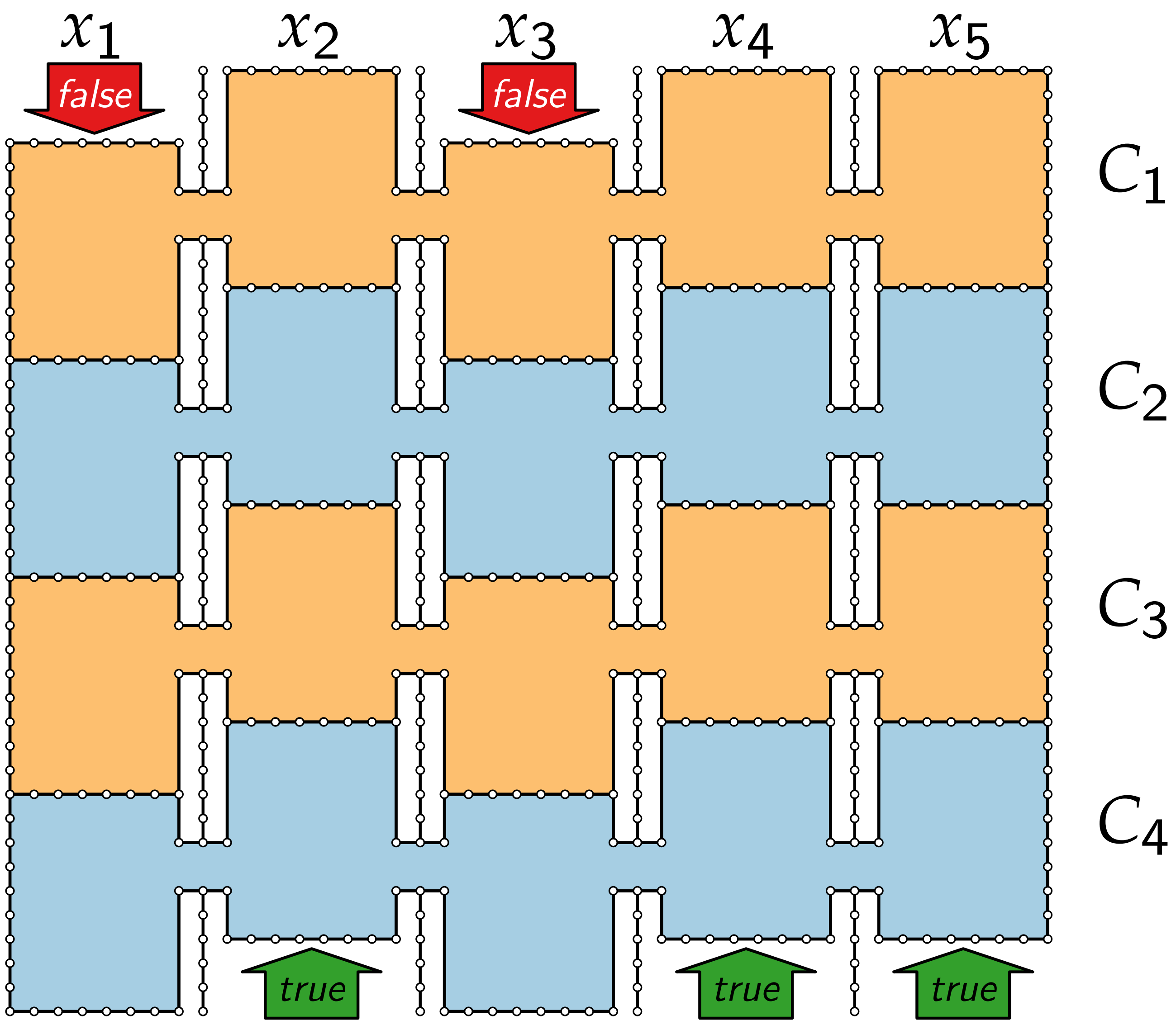
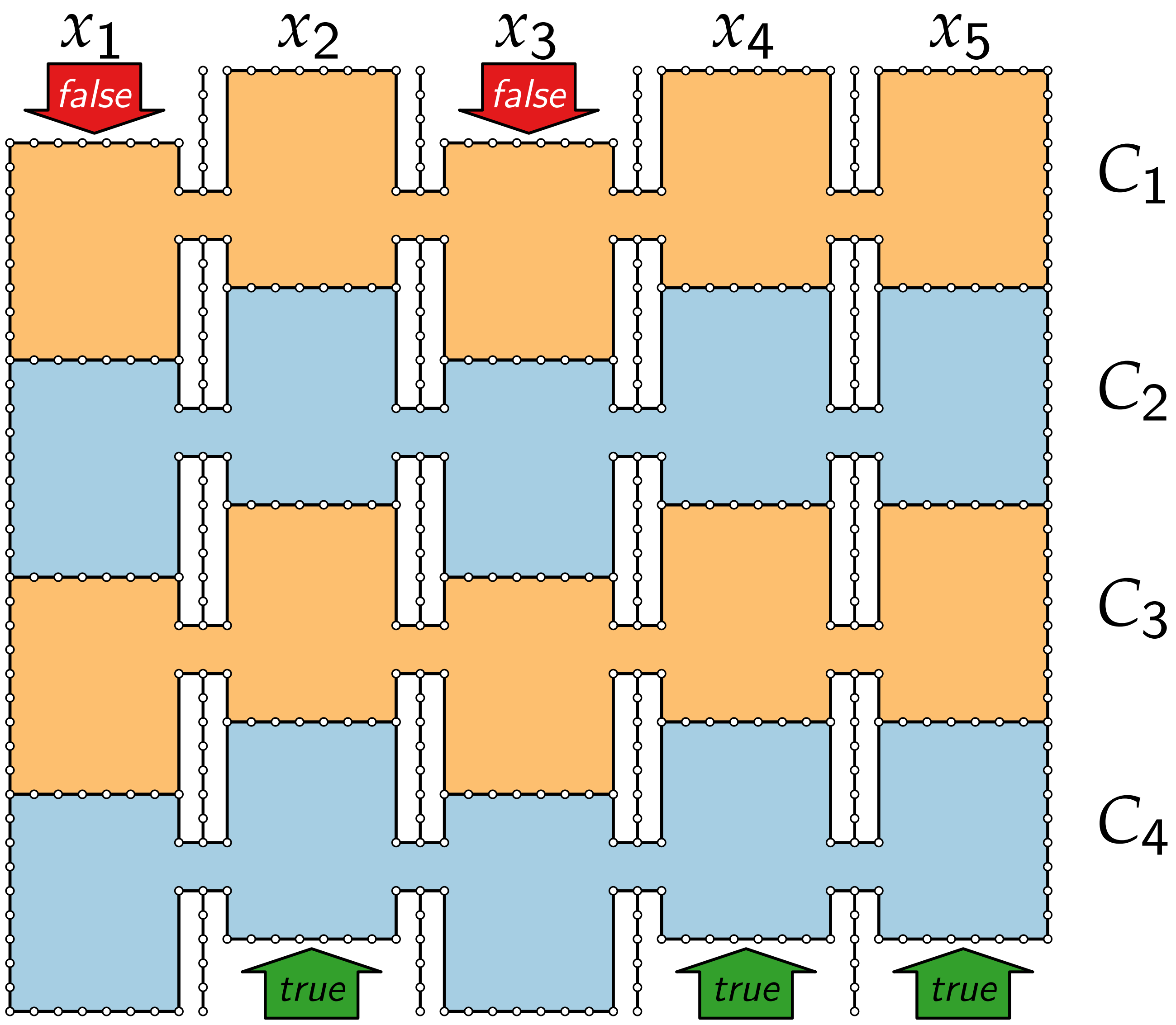


Clause gadgets

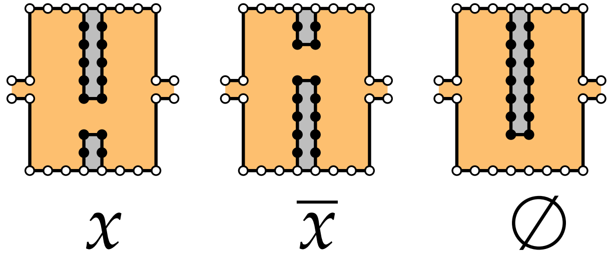


Clause gadgets



Example:

$C_1 = x_2 \vee \overline{x_4}$
$C_2 = x_1 \vee x_2 \vee \overline{x_3}$
$C_3 = x_5$
$C_4 = x_4 \vee \overline{x_5}$



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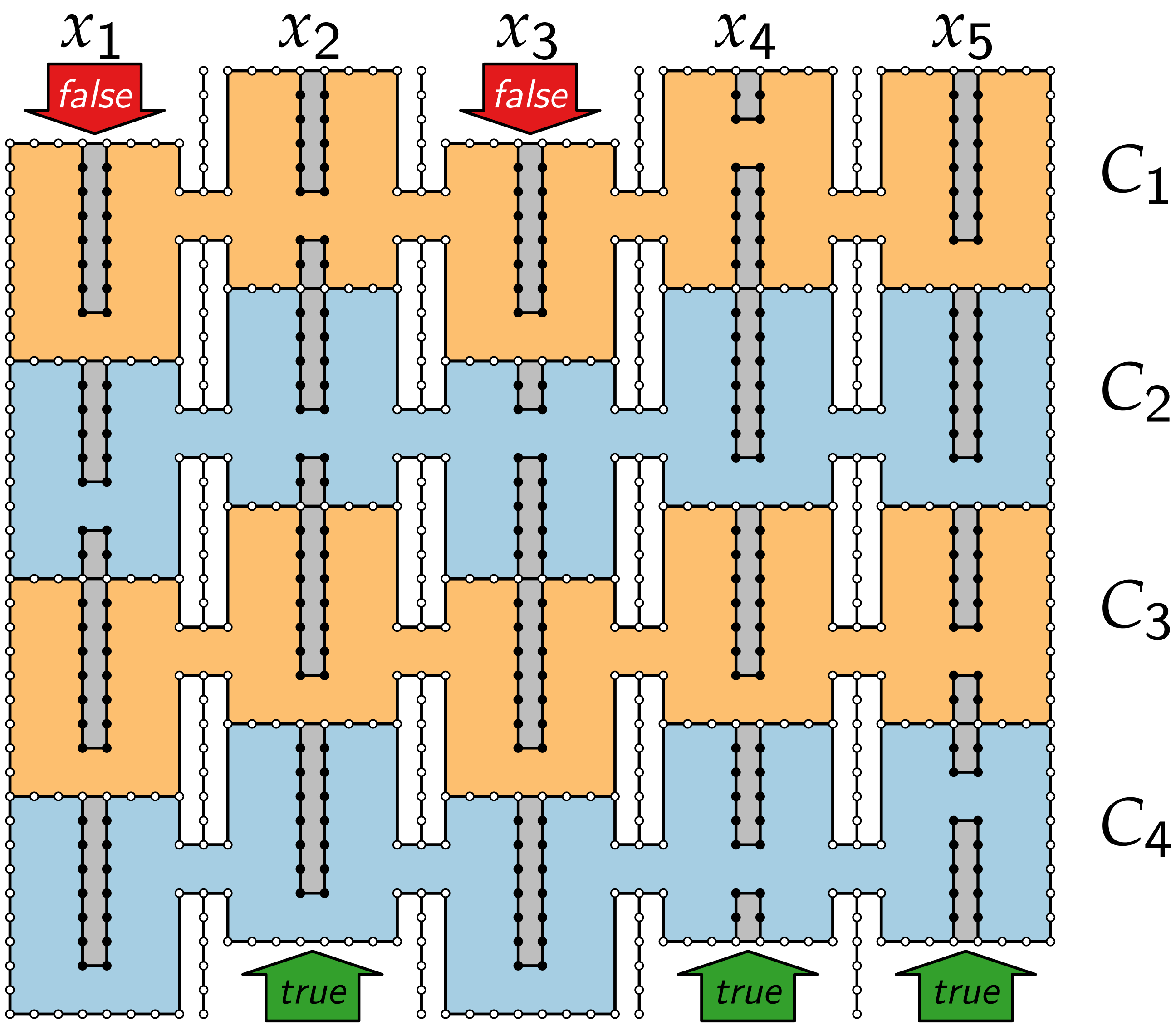

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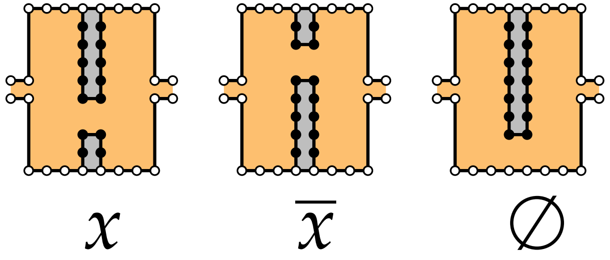
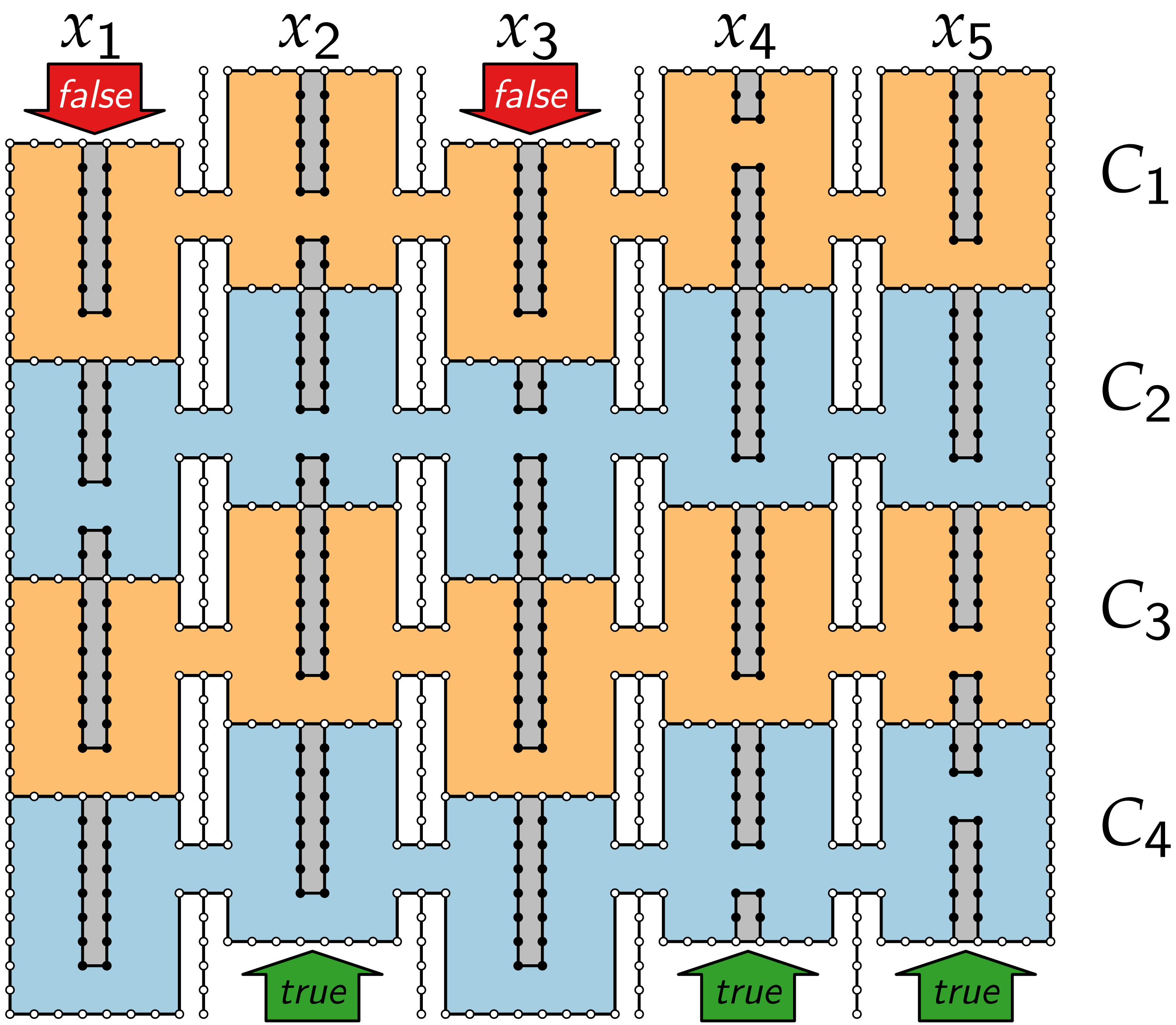


Clause gadgets



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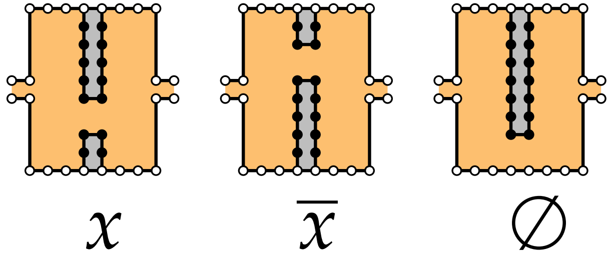
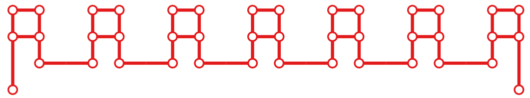
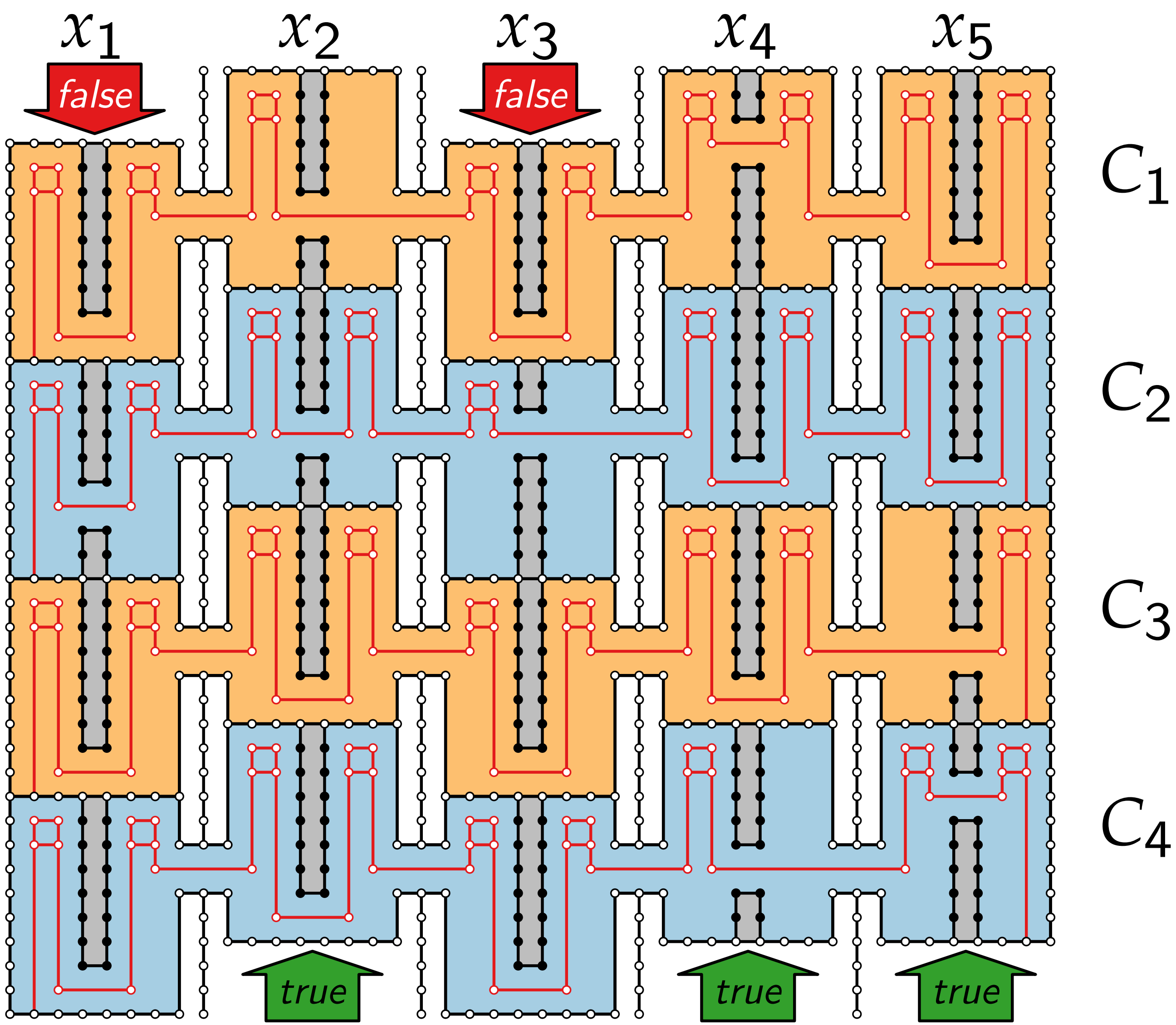




insert $(2n-1)$-chain through each clause

Clause gadgets



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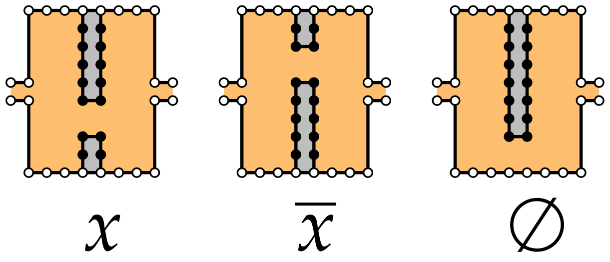

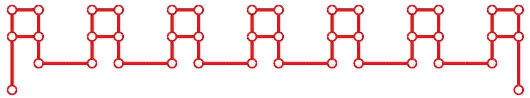

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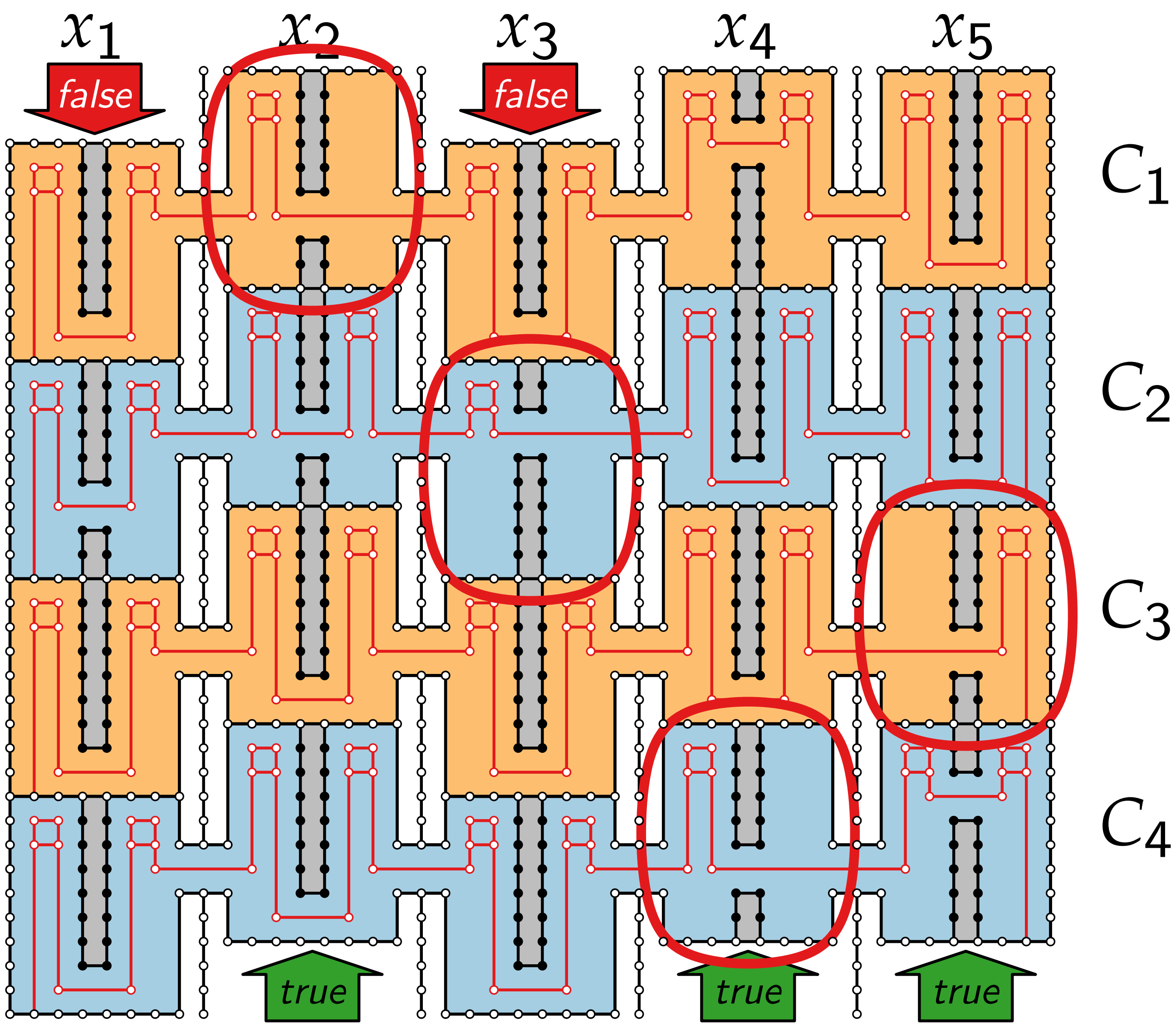

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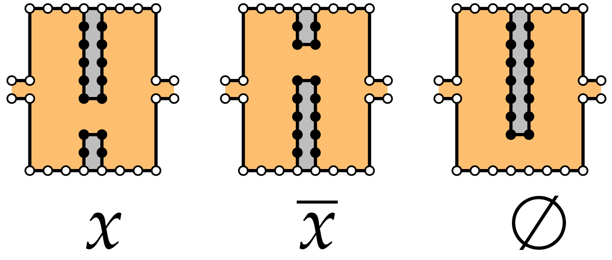

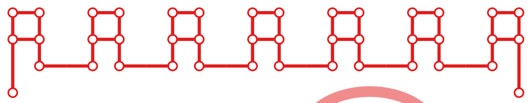

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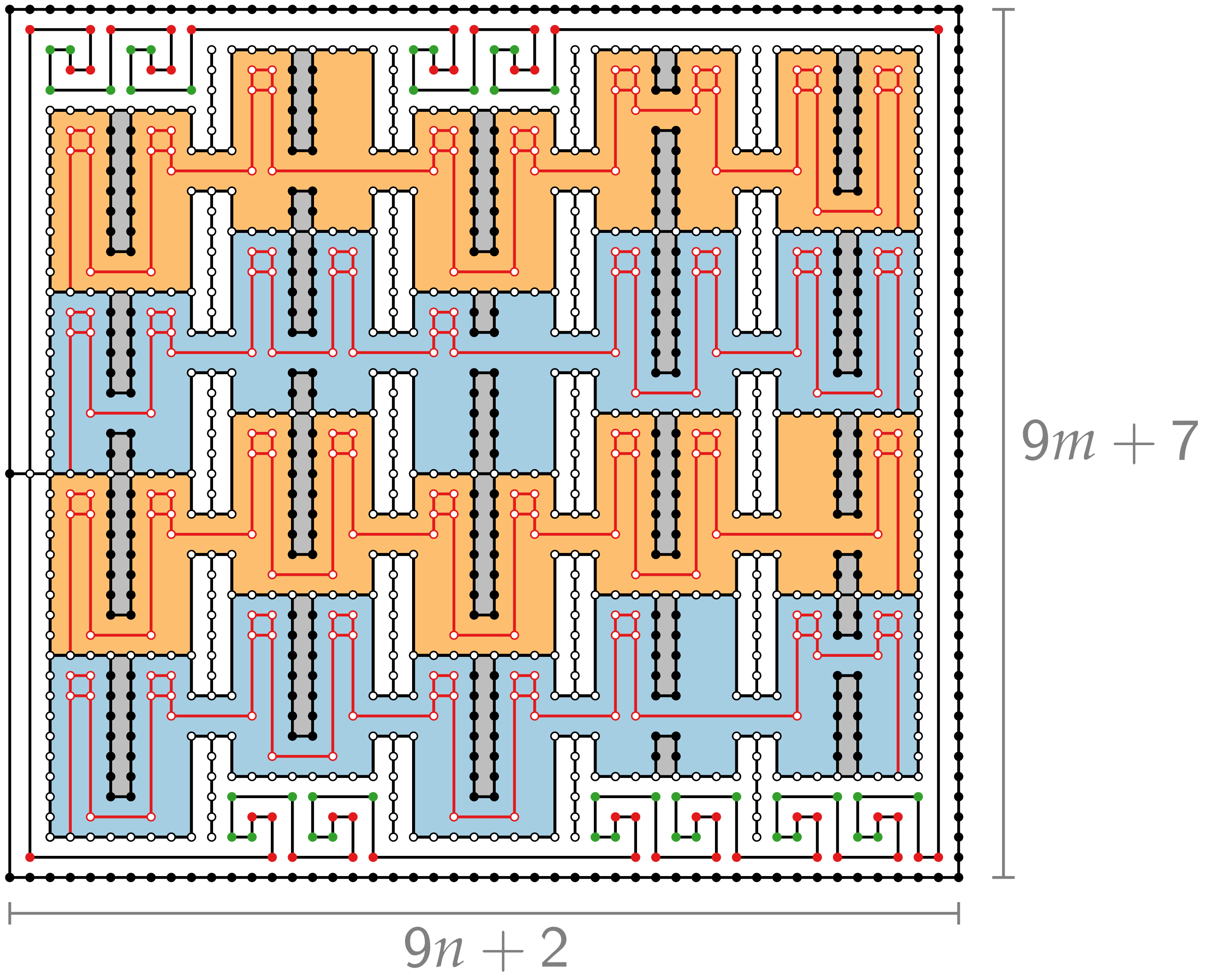
Complete reduction



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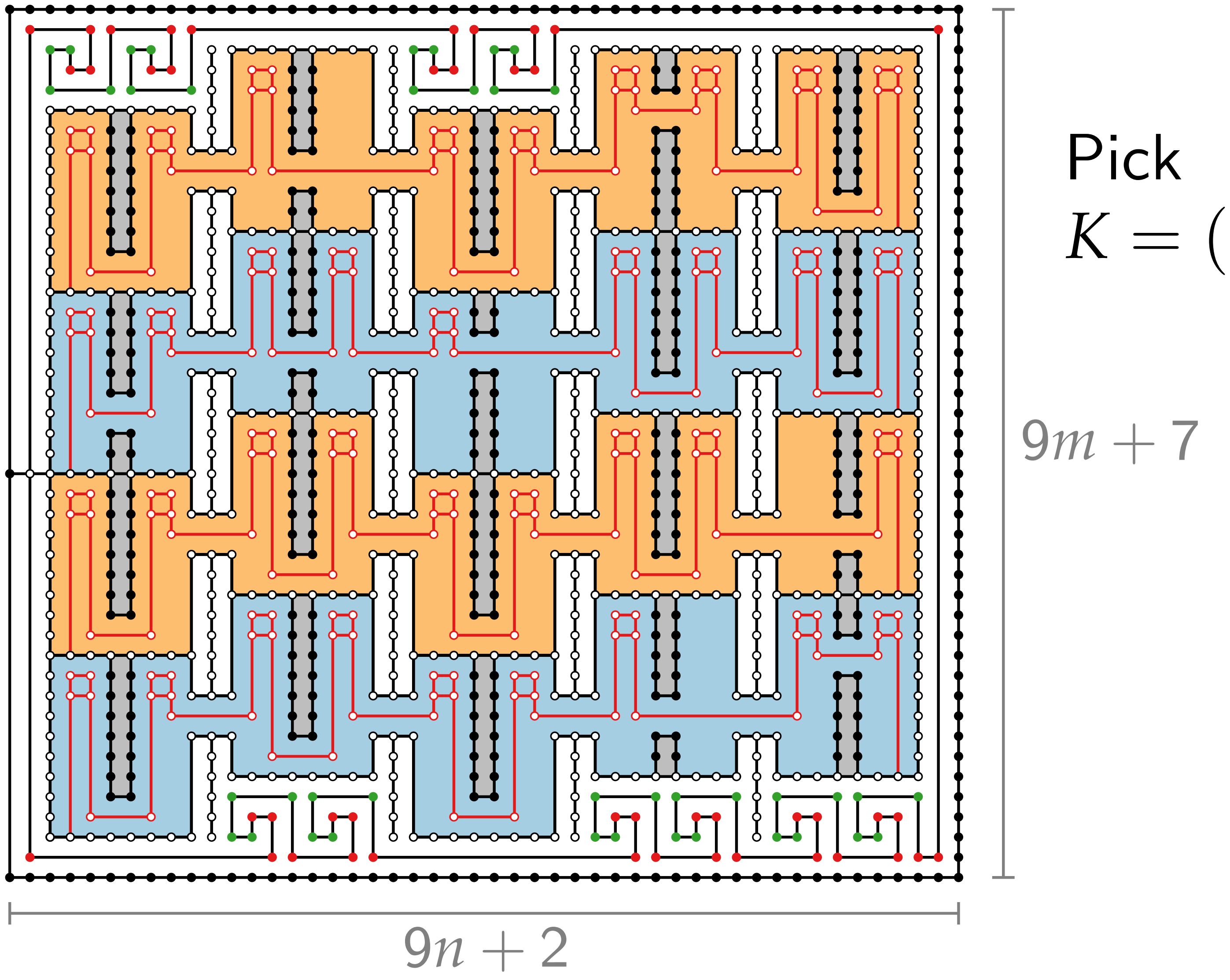
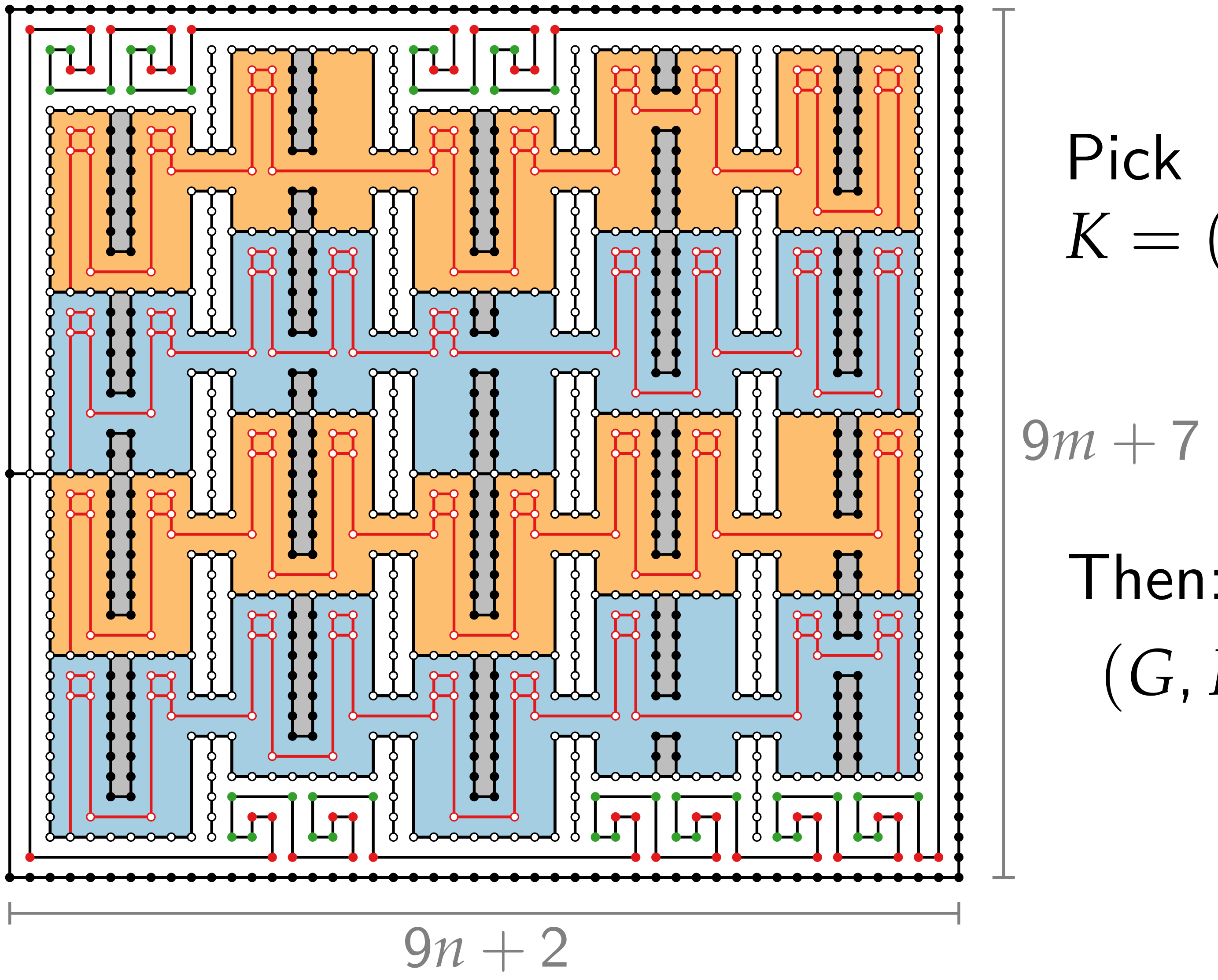


Pick
$K = (9n+2) \cdot (9m+7)$

Complete reduction



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$K = (9n + 2) \cdot (9m + 7)$

Then:

(G, H) has an area K drawing
$\Leftrightarrow$
Φ satisfiable $\square$

Literature

- [GD Ch. 5] for detailed explanation.
- [NR04, Ch. 8] T. Nishizeki, M. S. Rahman, "Planar Graph Drawing", Lecture Notes Series on Computing, Vol. 12, World Scientific, 2004.
  – Another detailed presentation.
- [Tam87] Tamassia "On embedding a graph in the grid with the minmum number of bends", 1987.
  – original paper on flow for bend minimisation
- [Bield96] T.C. Biedl, "Optimal orthogonal drawings of triconnected plane graphs", SWAT, 1996.
  –Upper bound on the area of a compact orthogonal drawing.
- [Pat01] Patrignani "On the complexity of orthogonal compaction", 2001.
  – NP-hardness proof of compactification