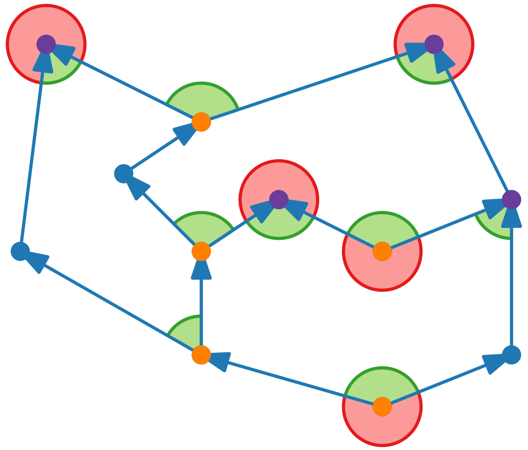
Visualisation of graphs

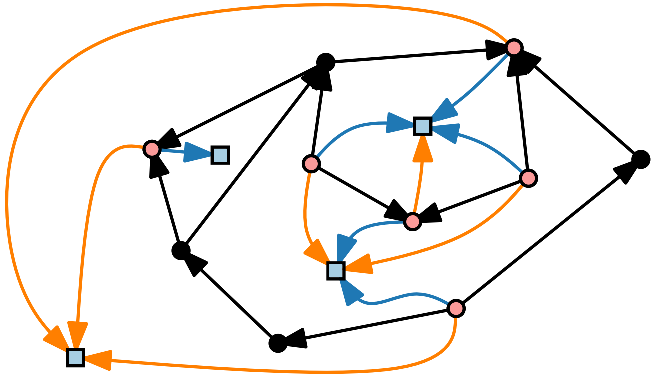
Upward planar drawings

Flow methods

Antonios Symvonis · Chrysanthi Raftopoulou

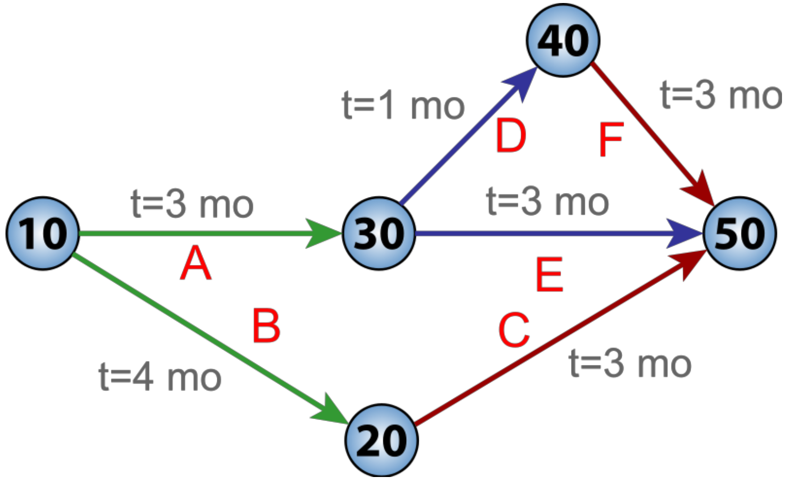
Fall semester 2022



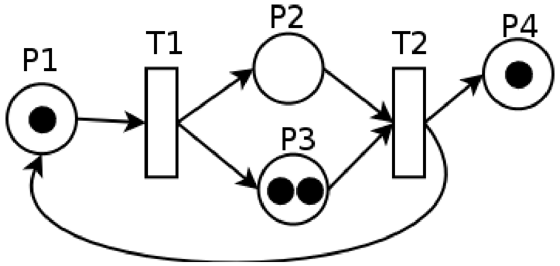


Upward planar drawings – motivation

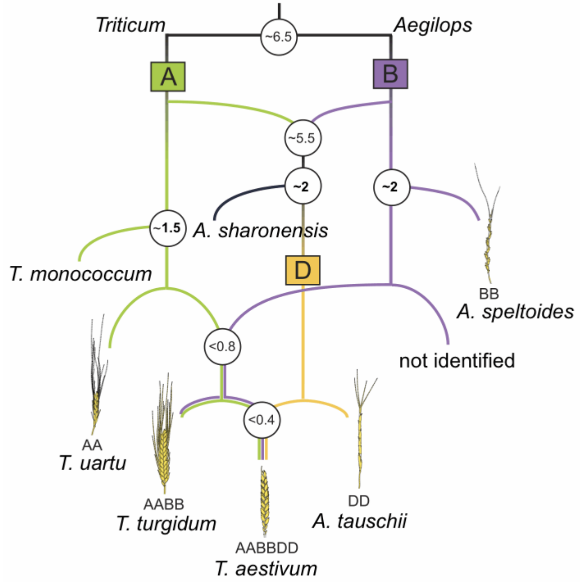
- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchie
  - ...



PERT diagram



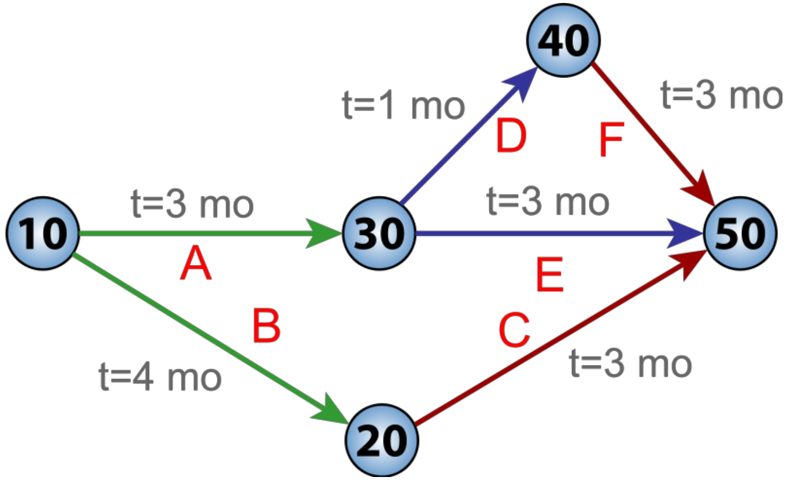
Petri net



Phylogenetic network

Upward planar drawings – motivation

- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchie
  - . . .
- Would be nice to have general direction preserved in drawing.



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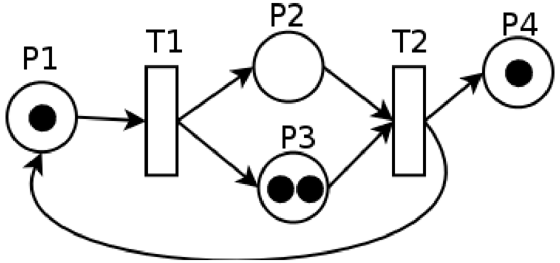

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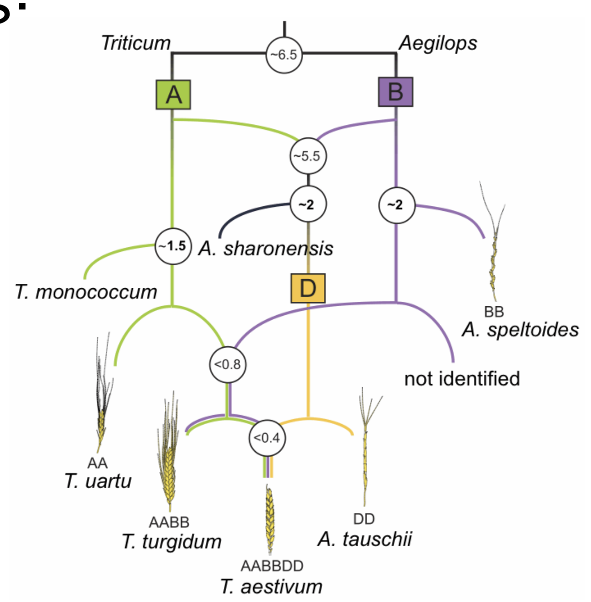
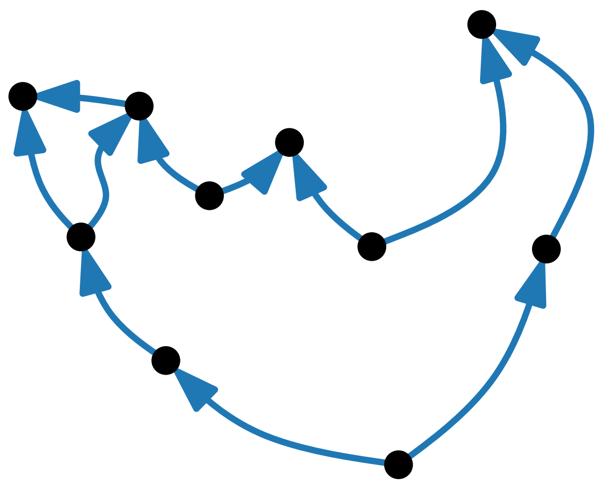


Phylogenetic network

Upward planar drawings – definition

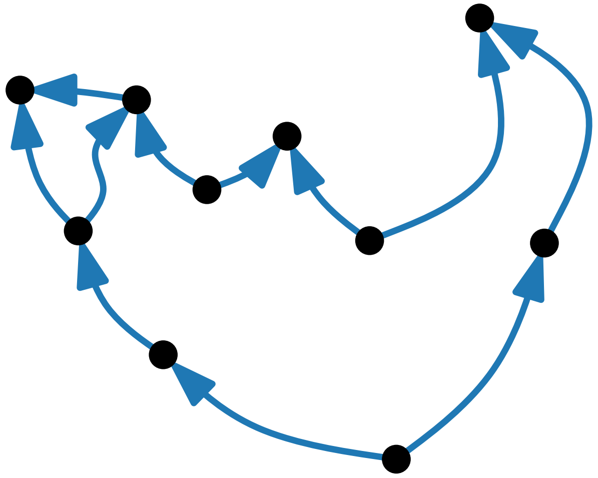
Definition.
A directed graph $G = (V, E)$ is **upward planar** when it admits a drawing Γ (vertices = points, edges = simple curves) that is

- planar and
- where each edge is drawn as an upward, y-monotone curve.



Upward planarity – necessary conditions

- For a digraph G to be upward planar, it has to be:
  - planar



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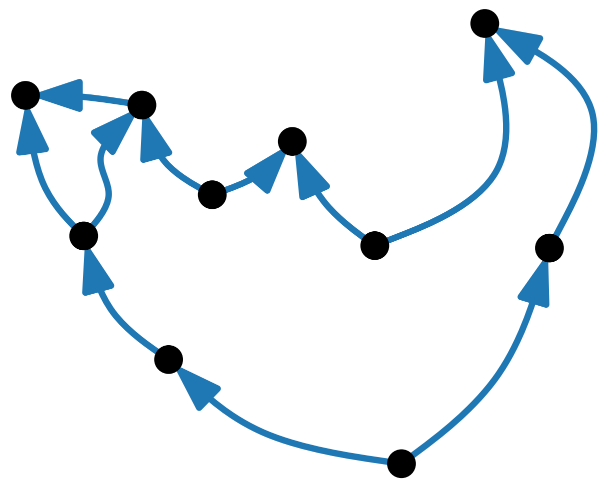

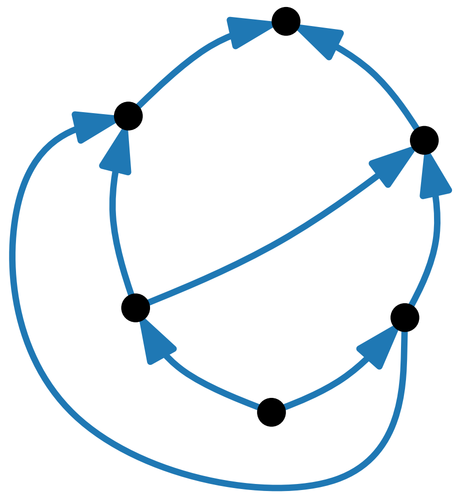

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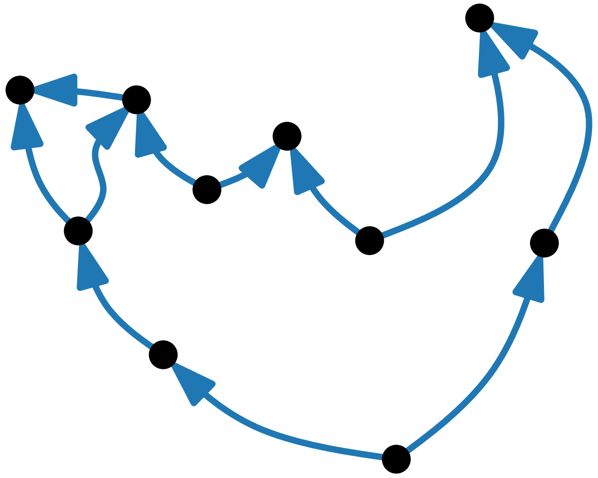

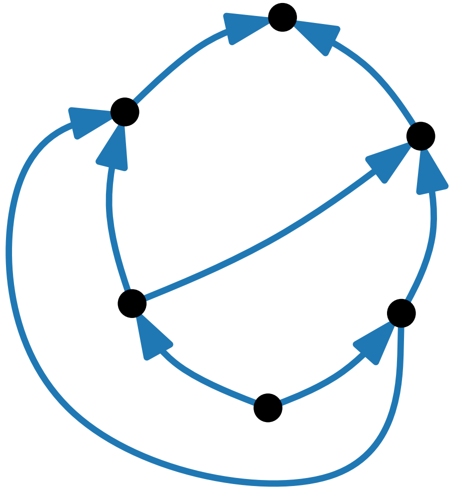

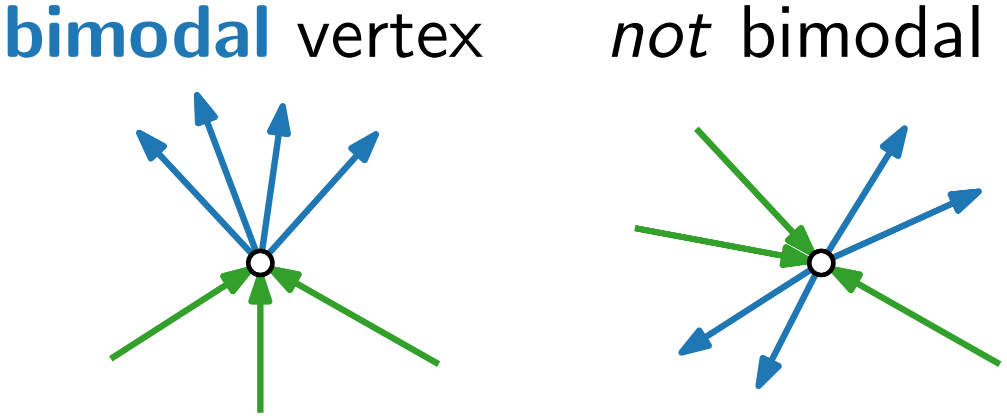

Upward planarity – necessary conditions

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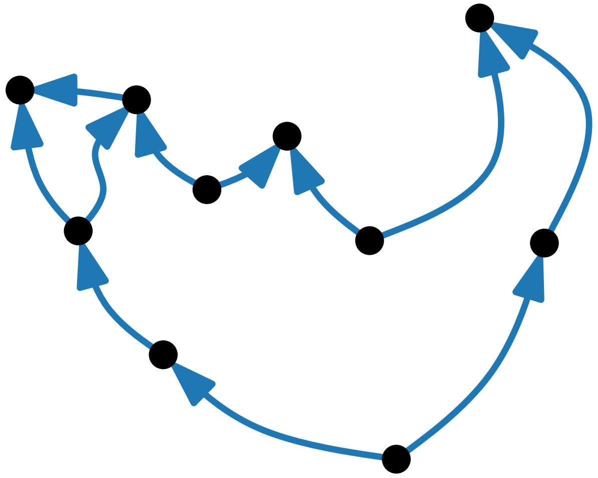
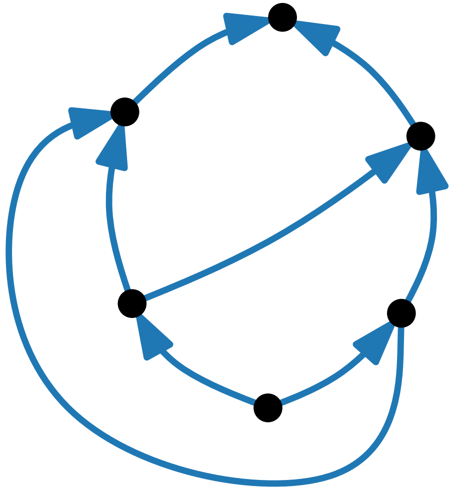
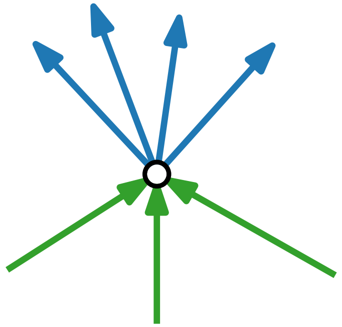
  - planar
  - acyclic
  - bimodal

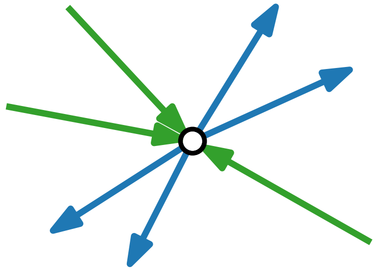




bimodal vertex

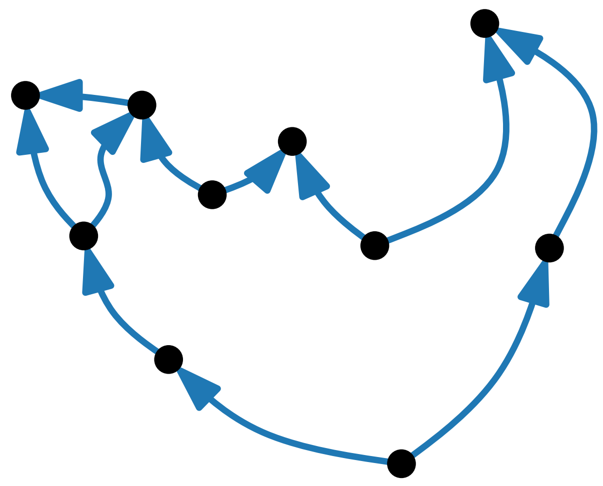


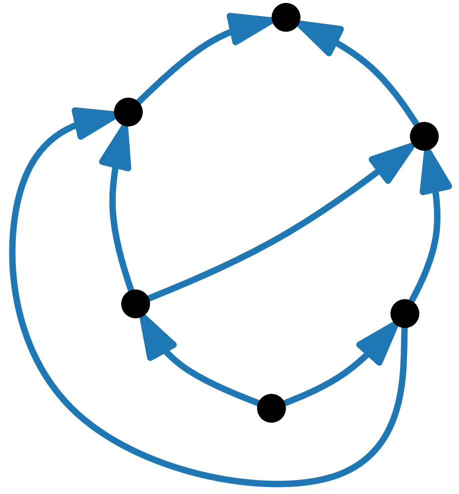
not bimodal



Upward planarity – necessary conditions

- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic
  - bimodal
- . . . but these conditions are *not sufficient*.





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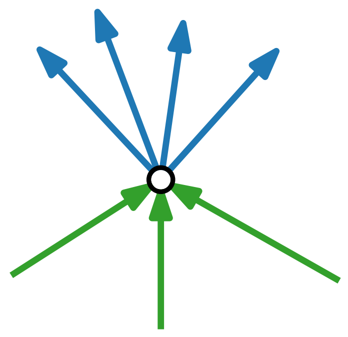

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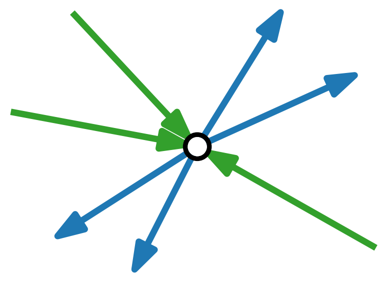

Upward planarity – characterisation

Theorem 1. [Kelly 1987, Di Battista, Tamassia, 1988, see GD Ch. 6]
For a digraph G the following statements are equivalent:

1. G is upward planar.
2. G admits an upward planar straight-line drawing.
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$\underbrace{}$ no crossings

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Embedded such that s and t are on the outerface f_0.

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Additionally:
Embedded such that s and t are on the outerface f_0.
or:
Edge (s, t) exists.

no crossings

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(2) $\Rightarrow$ **(1)** By definition.

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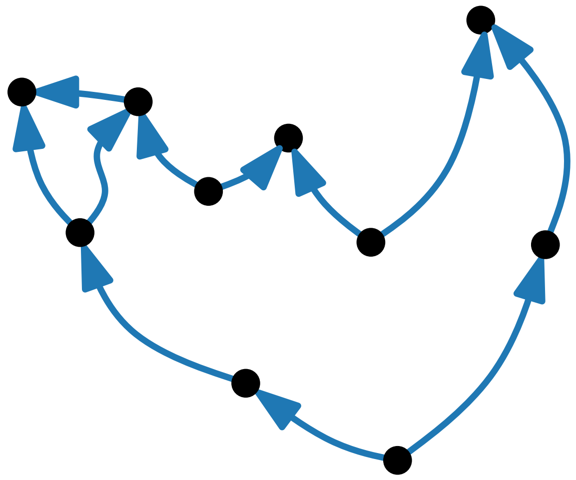

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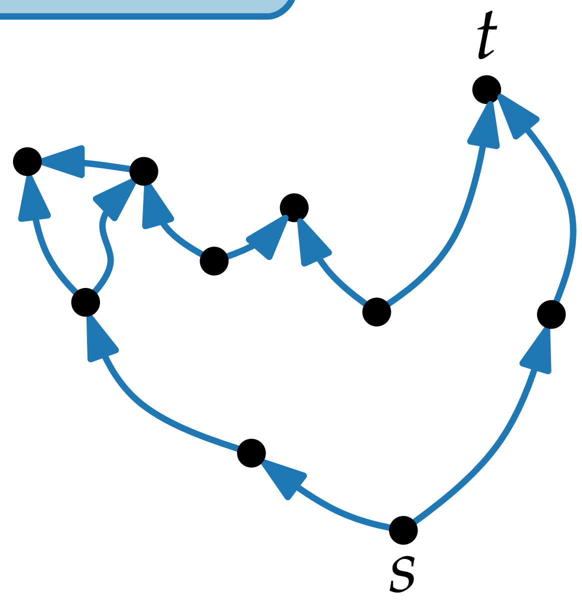

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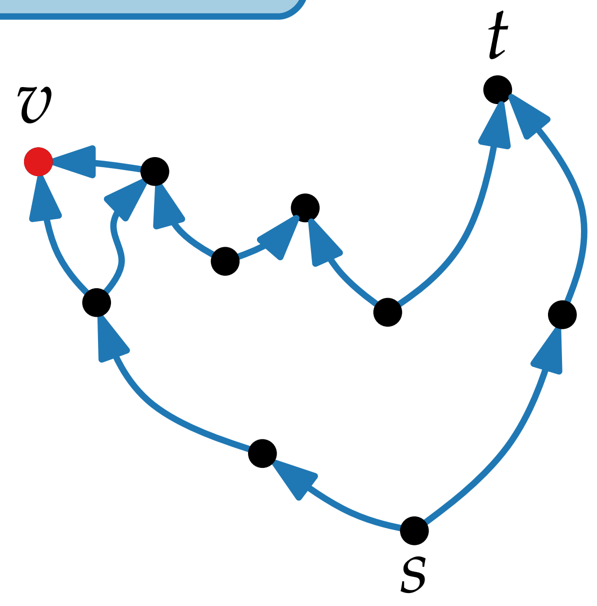

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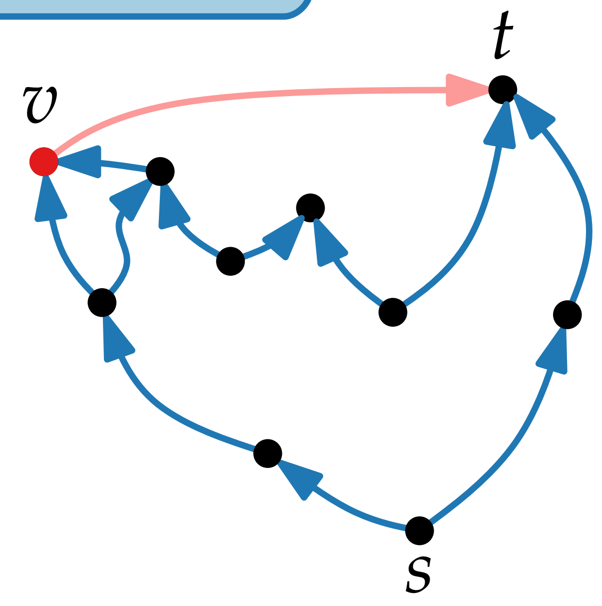

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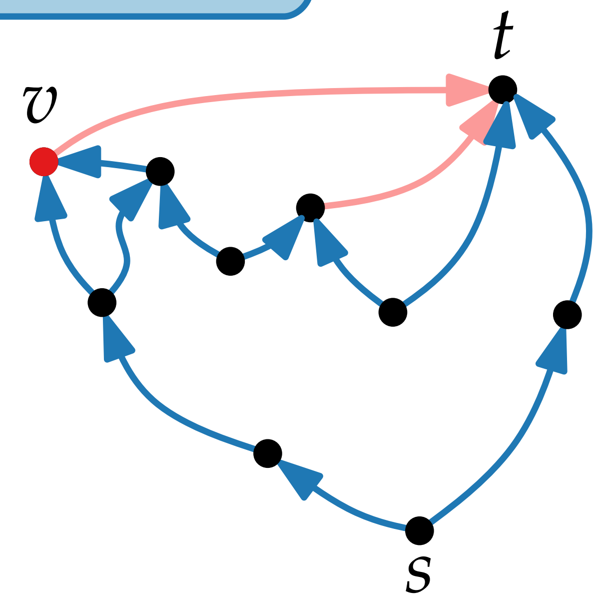

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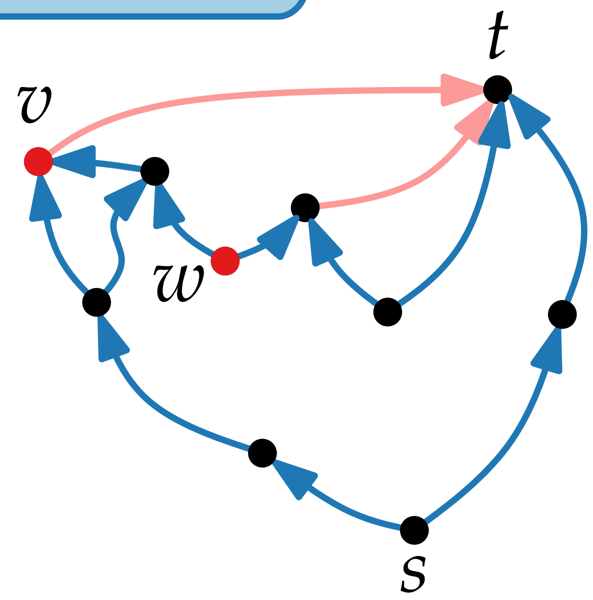

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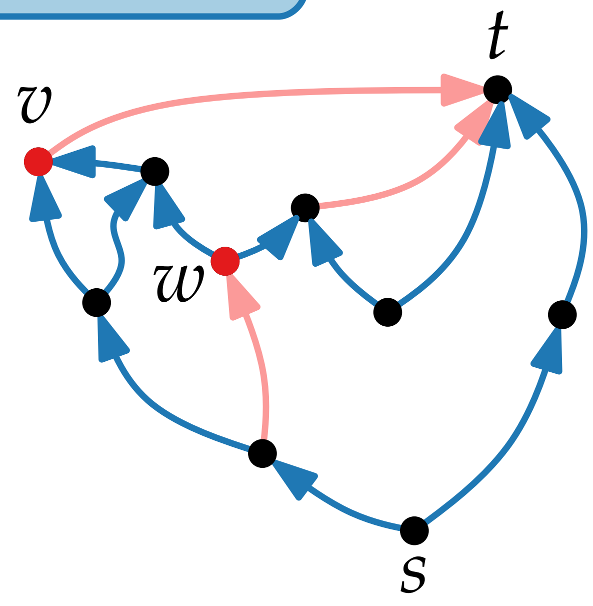

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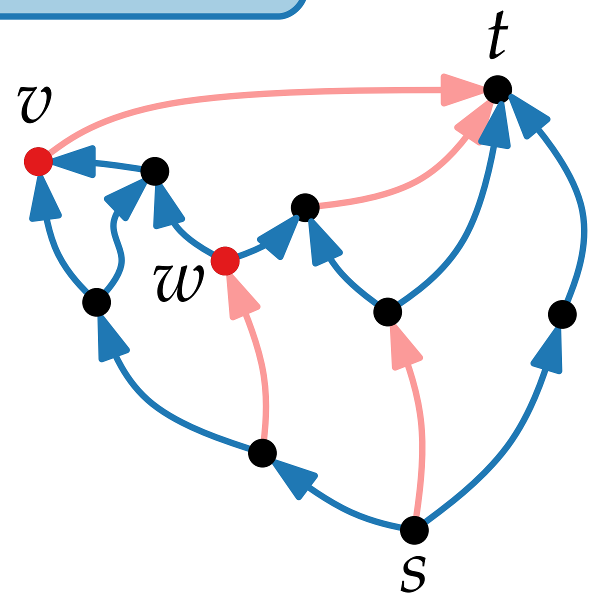

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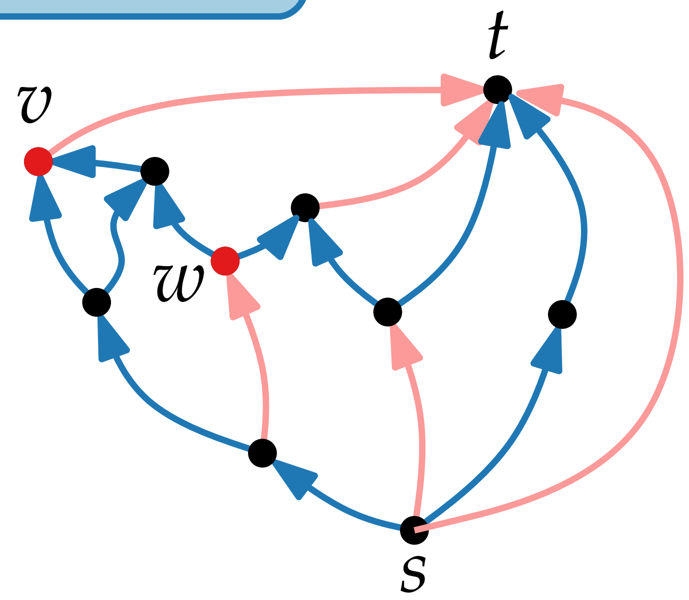

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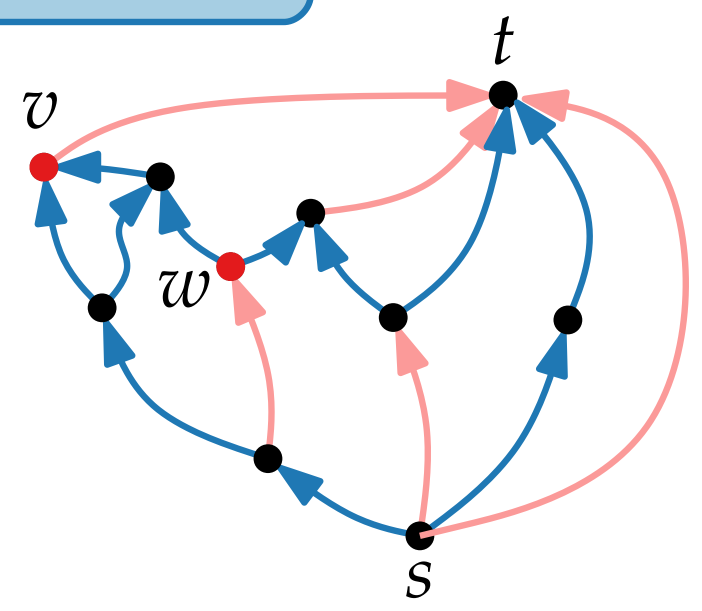

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(2) ⇒ (1) By definition. **(1) ⇔ (3)** Example:
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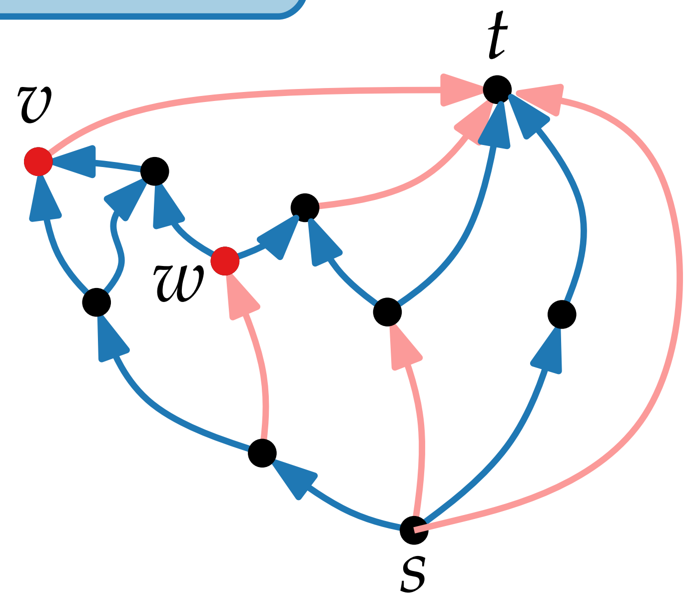
Claim.
Can draw in prespecified triangle.



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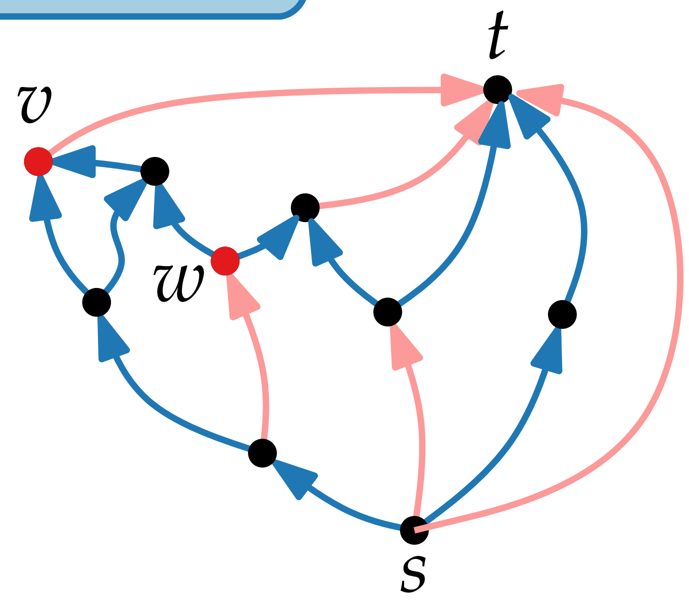

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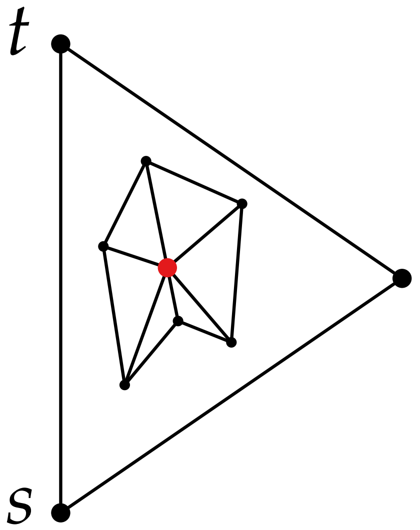

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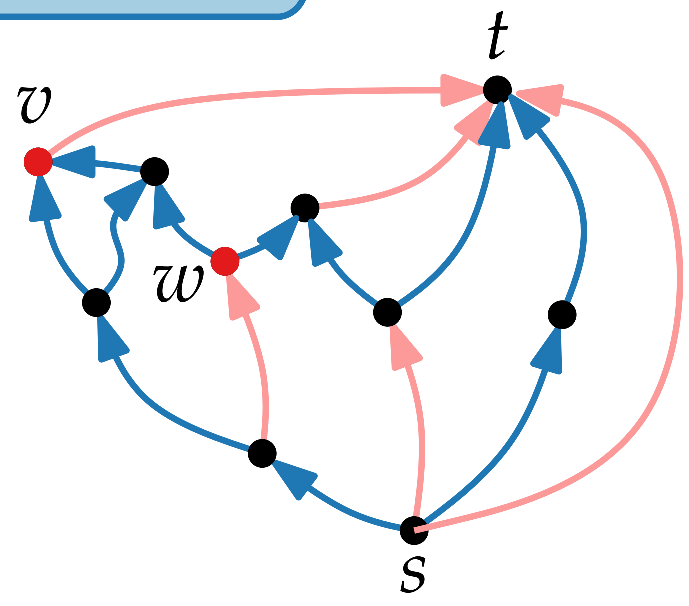

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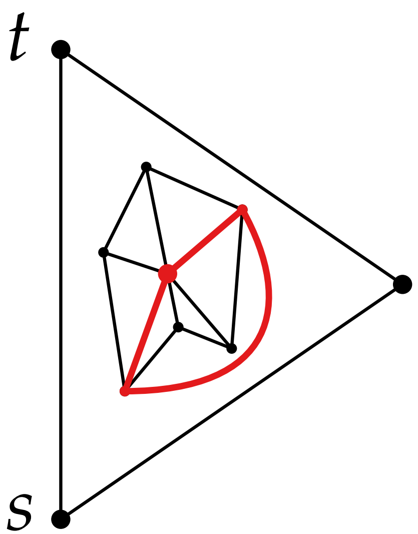

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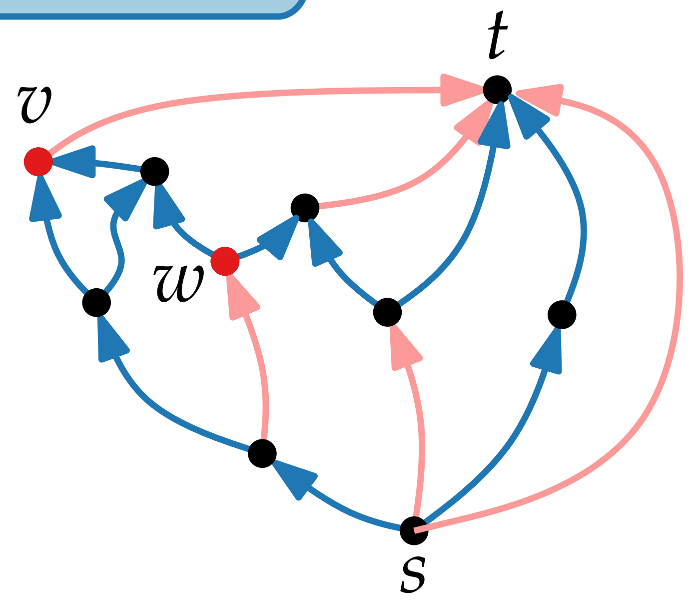

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Claim.
Can draw in prespecified triangle. Apply induction.

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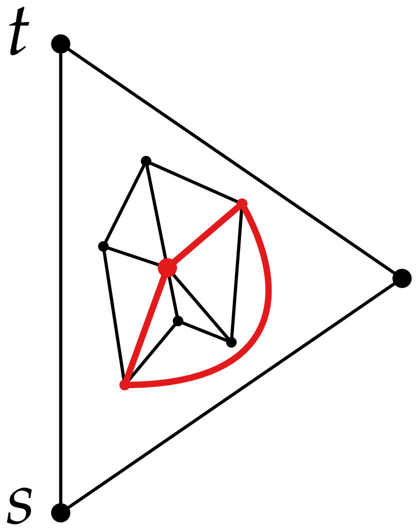

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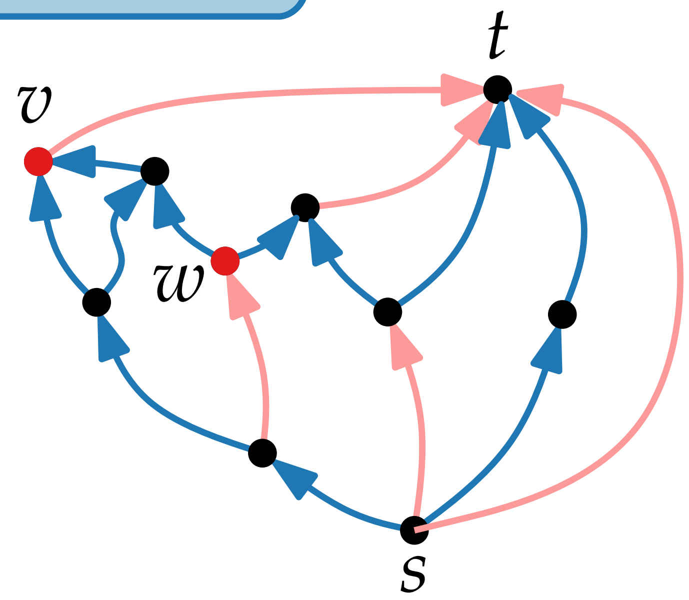

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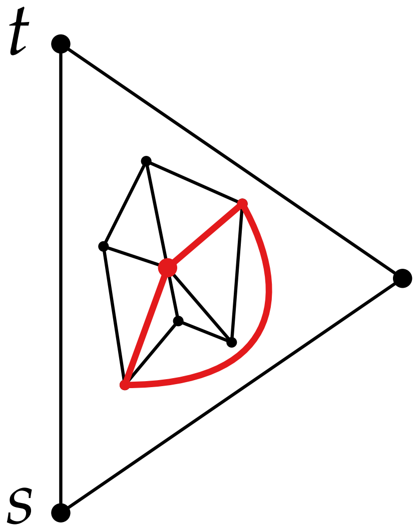

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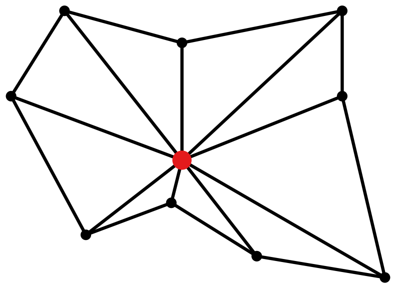

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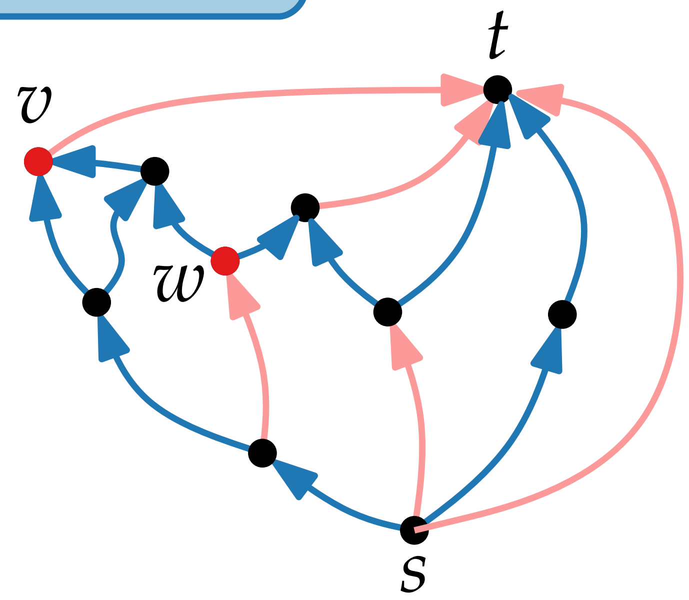

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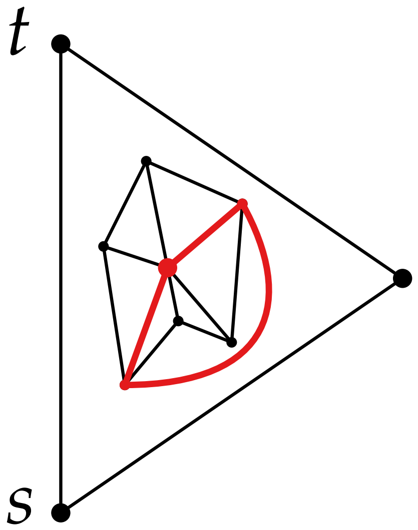

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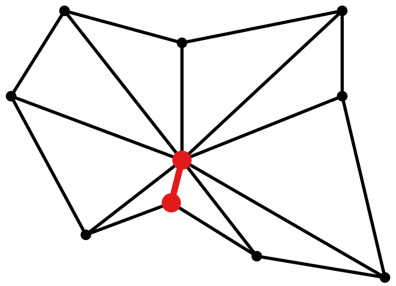

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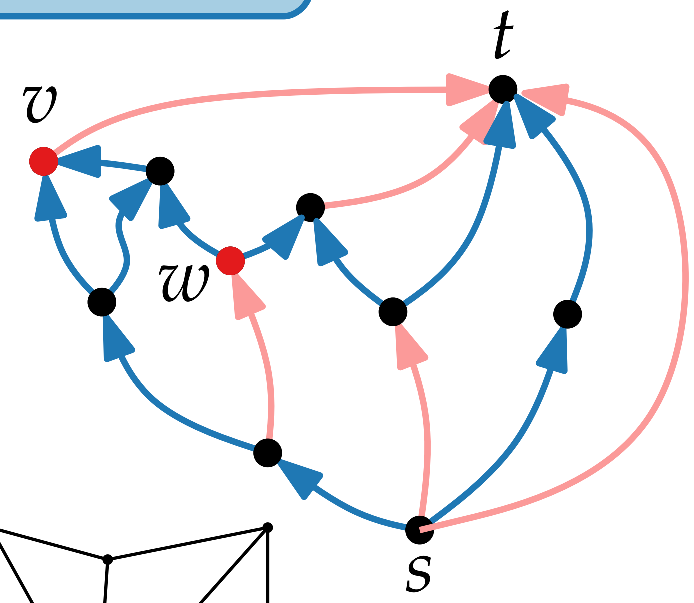

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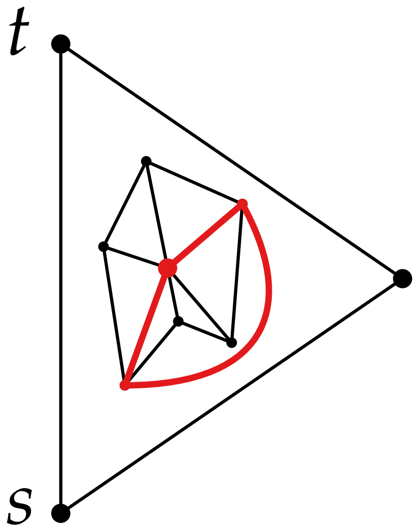

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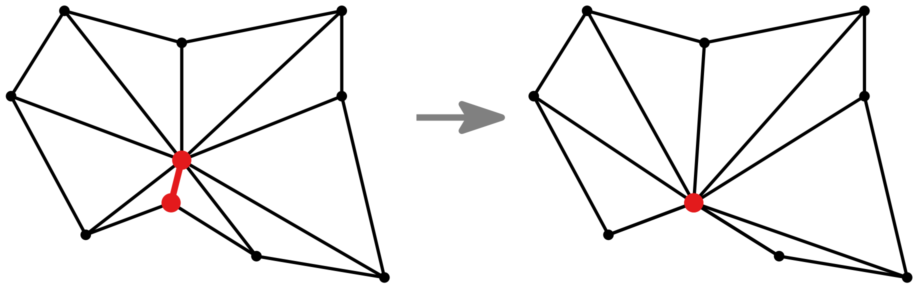
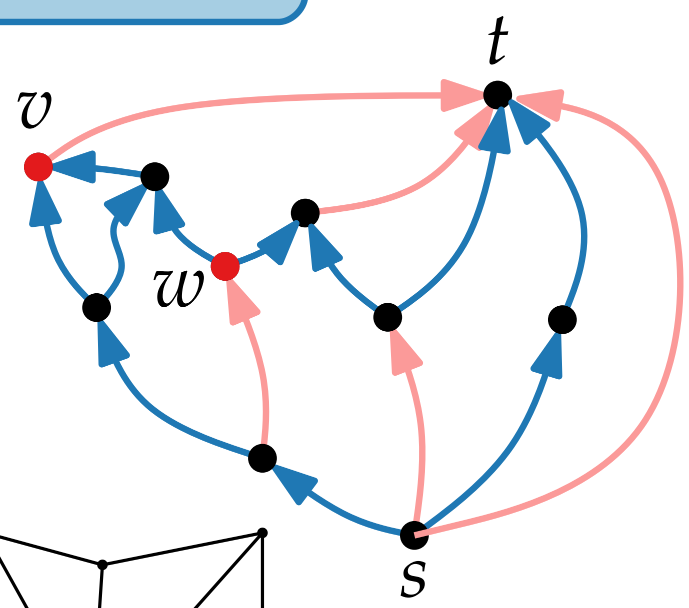


Upward planarity – characterisation

Theorem 1. [Kelly 1987, Di Battista, Tamassia, 1988, see GD Ch. 6]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

Proof.



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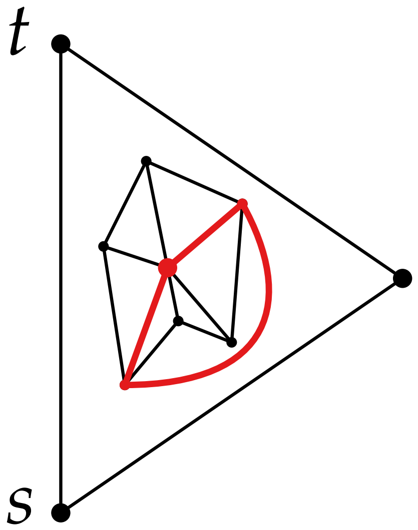

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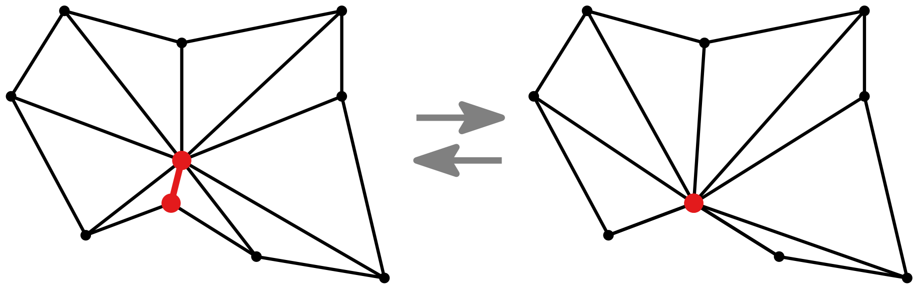

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Theorem. [Garg, Tamassia, 1995]
For a *planar acyclic* digraph it is in general NP-hard to decide whether it is upward planar.

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Theorem. [Hutton, Libow, 1996]
For a *single-source* acyclic digraph it can be tested in $\mathcal{O}(n)$ time whether it is upward planar.

The problem

Fixed embedding upward planarity testing.
Let $G = (V, E)$ be a plane digraph with the embedding given by the set of faces F and the outer face f_0.
Test whether G is upward planar (wrt to F, f_0).

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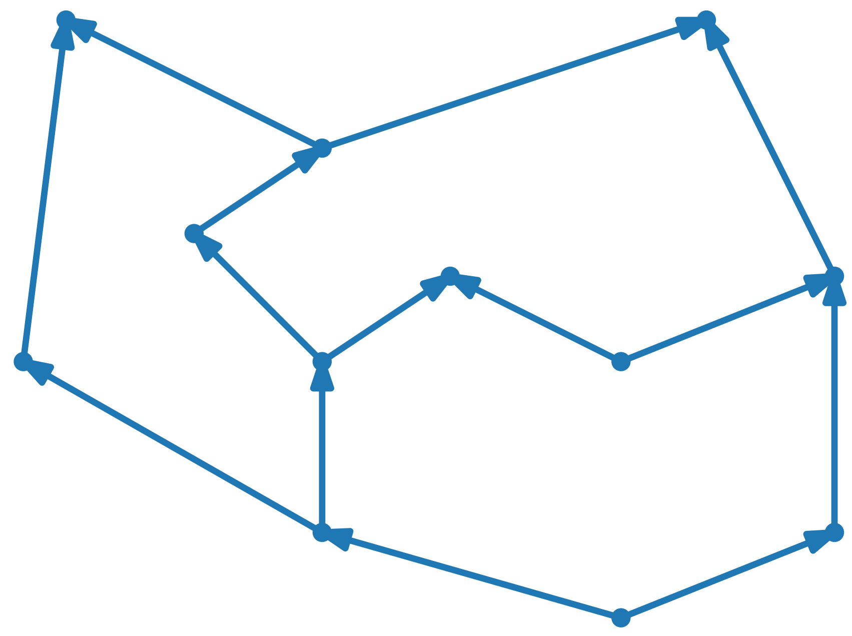
Idea.

- Find property that any upward planar drawing of G satisfies.
- Formalise property.
- Find algorithm to test property.

Angles, local sources & sinks

Definitions.



Angles, local sources & sinks

Definitions.

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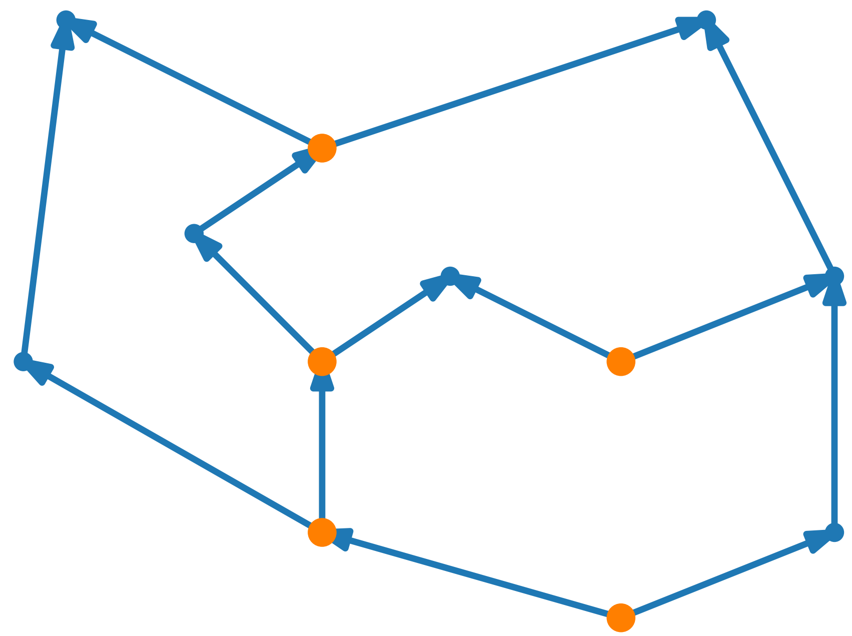

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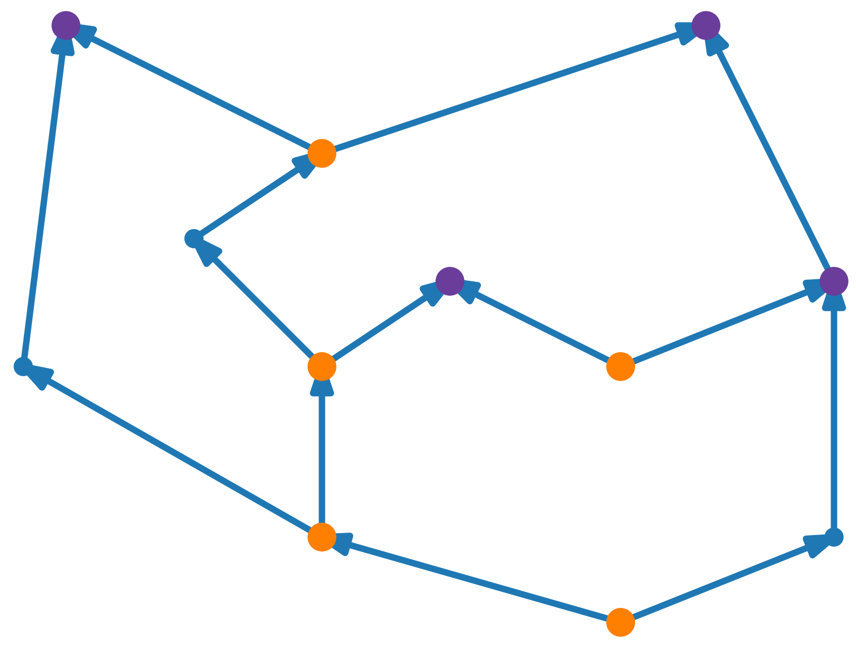

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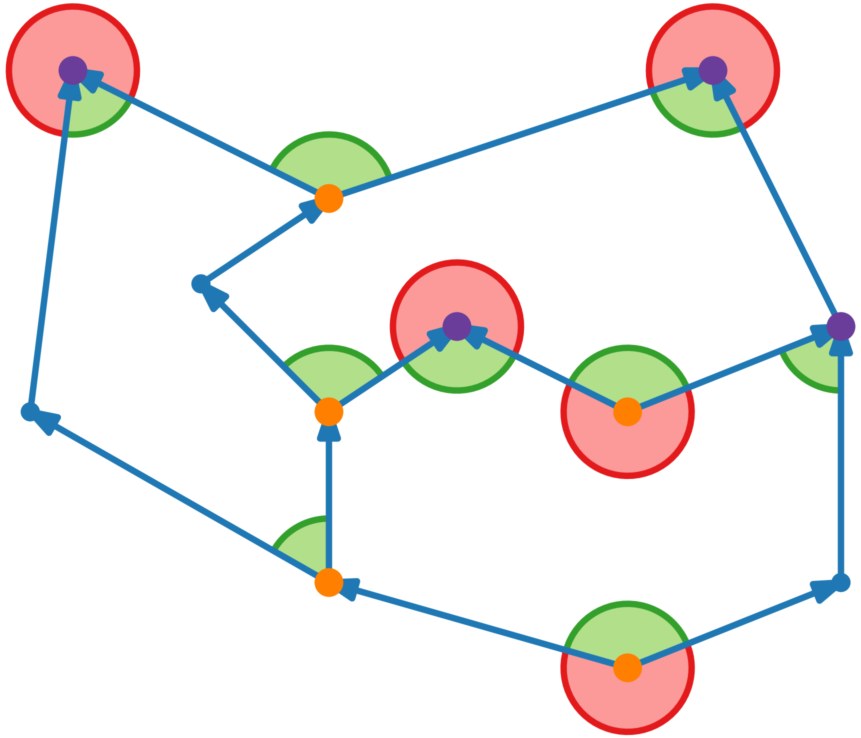
- $L(f) = \#$ large angles in f



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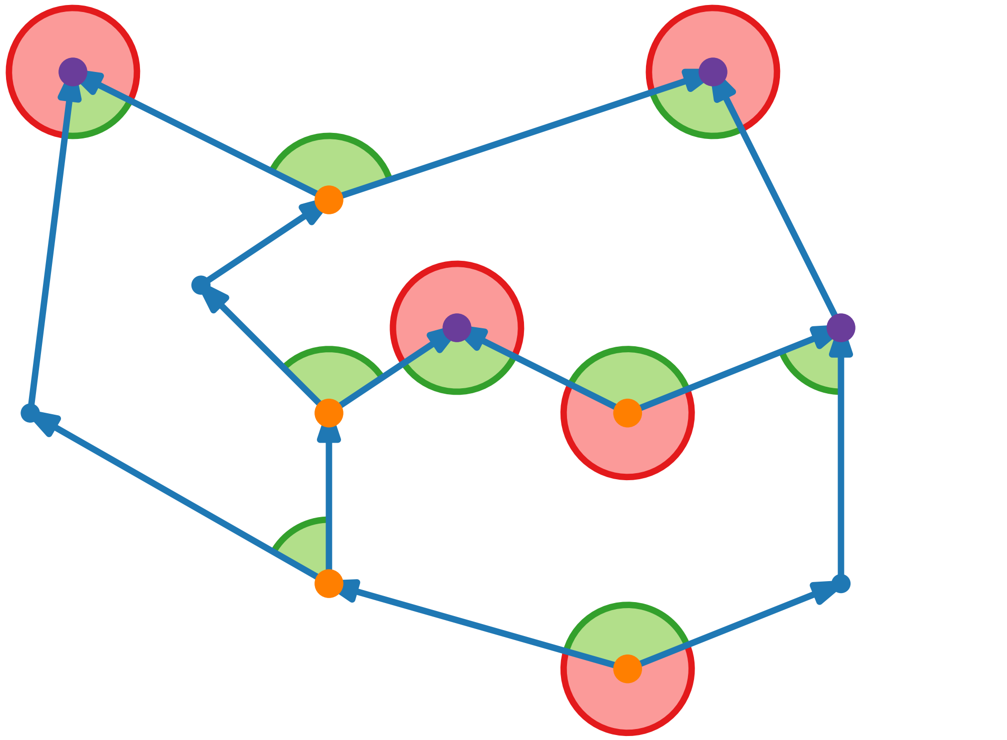

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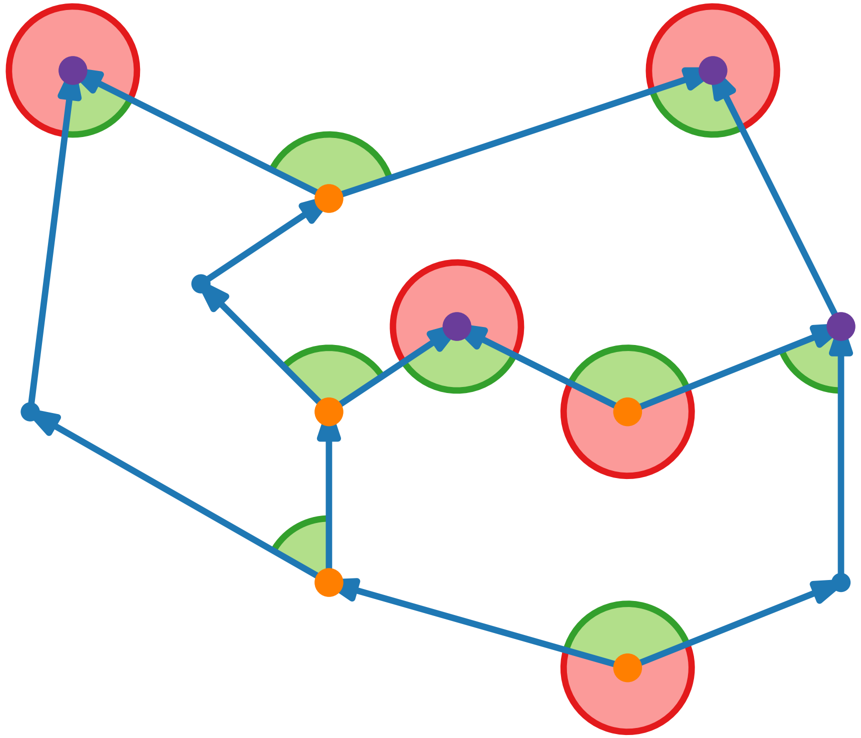

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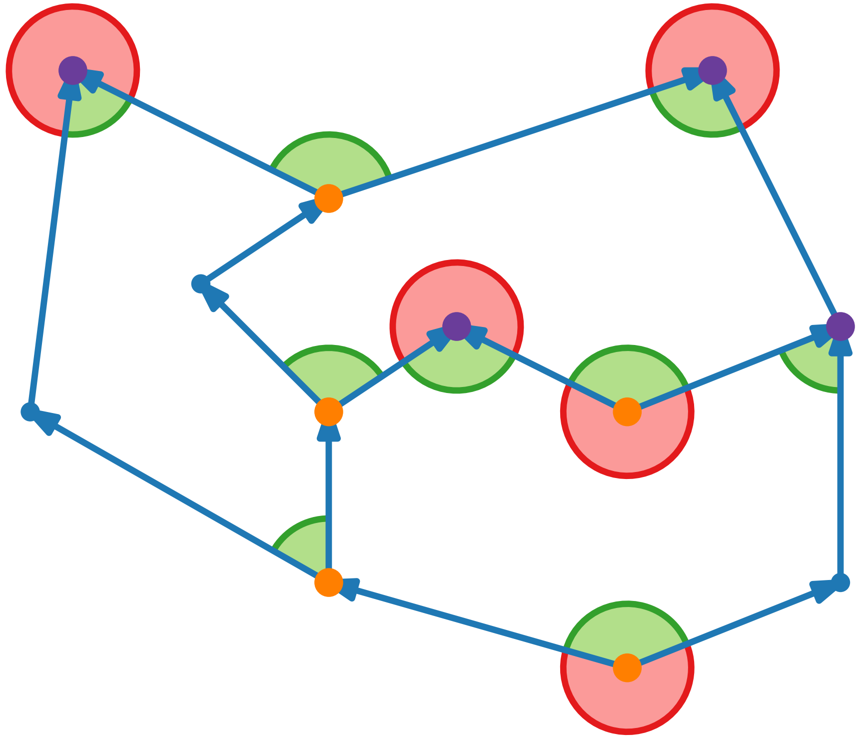
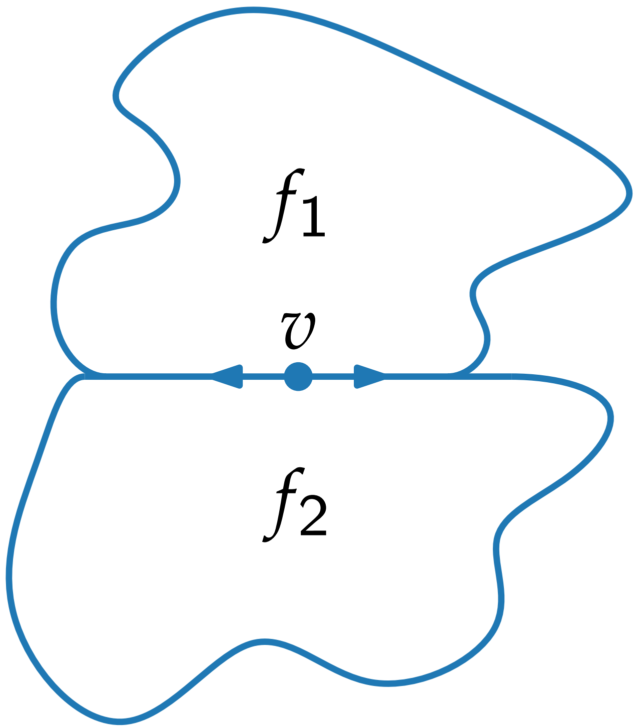


Lemma 1.
$L(f) + S(f) = 2A(f)$

Assignment problem

- Vertex v is a global source for f_1 and f_2.
- Has v a **large** angle in f_1 or f_2?



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Lemma 2.

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Proof by induction.

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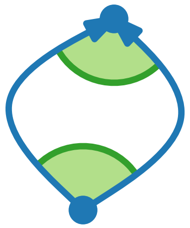

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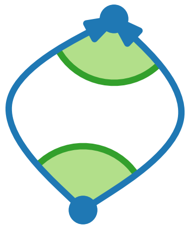
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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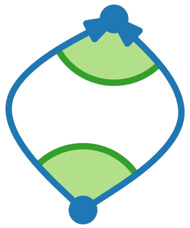
- $L(f) = 0$  $\Rightarrow S(f) = 2$
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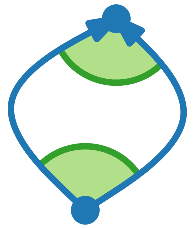

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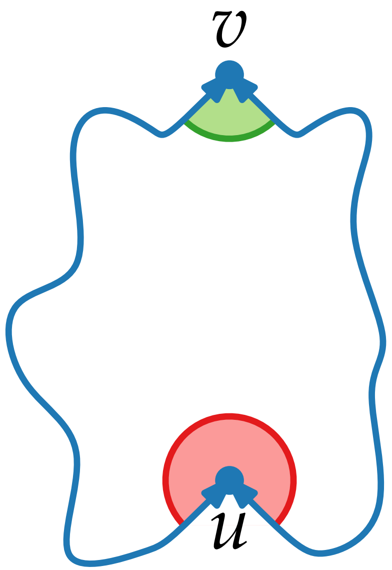

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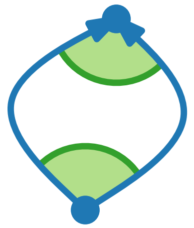
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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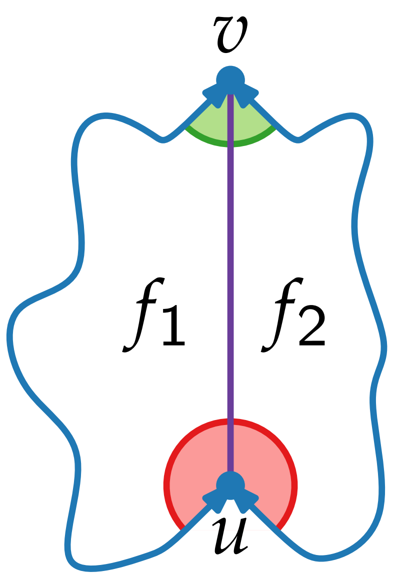

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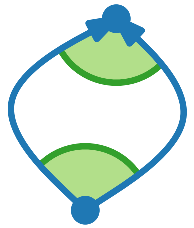
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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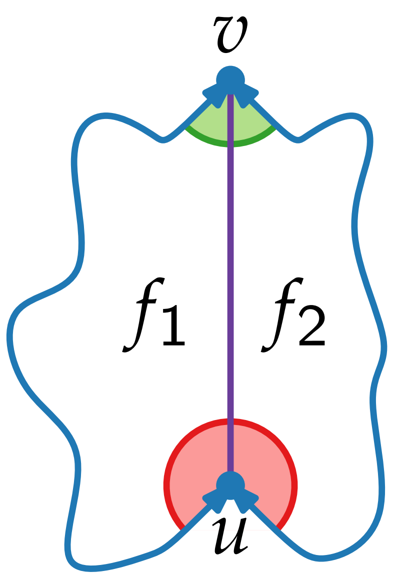

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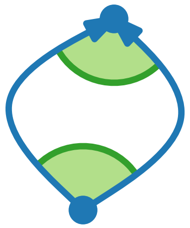
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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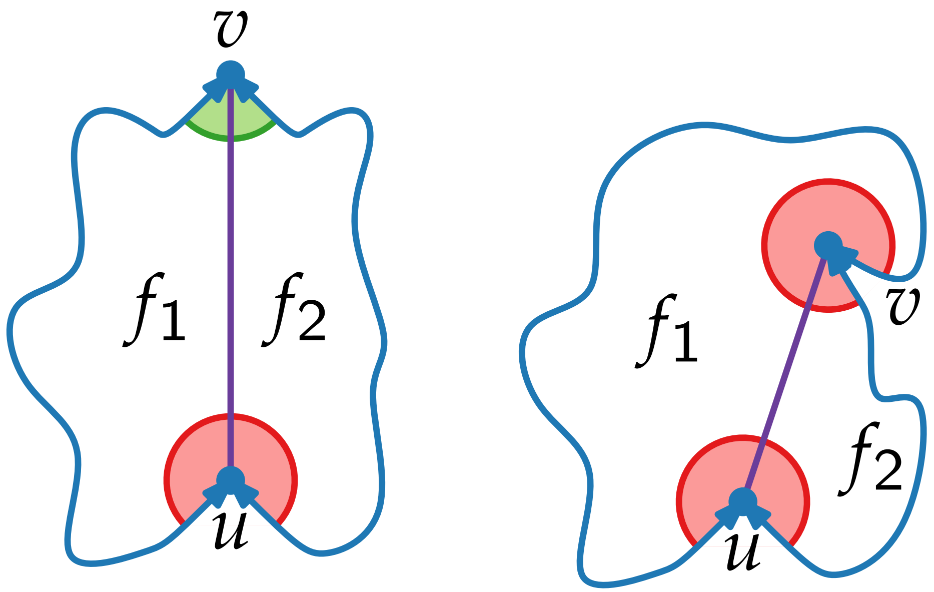

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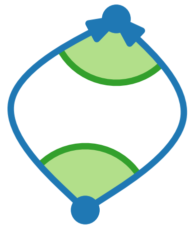
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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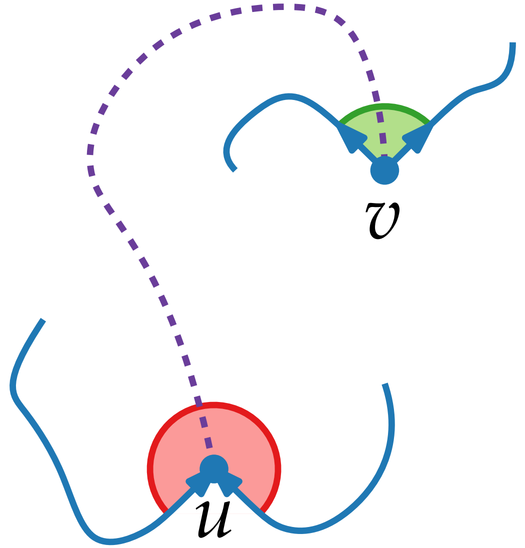

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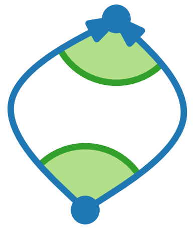

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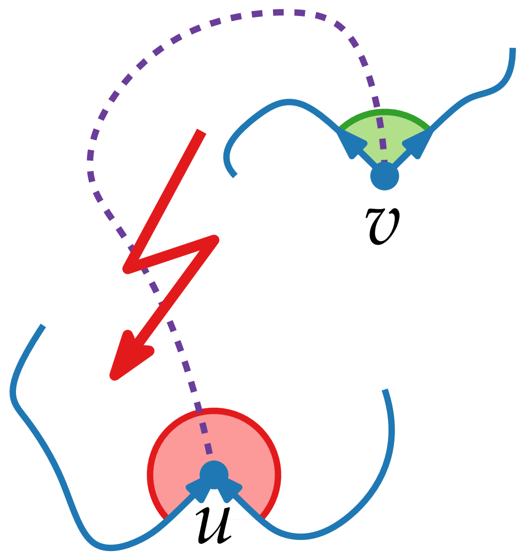

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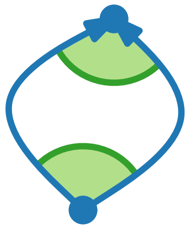
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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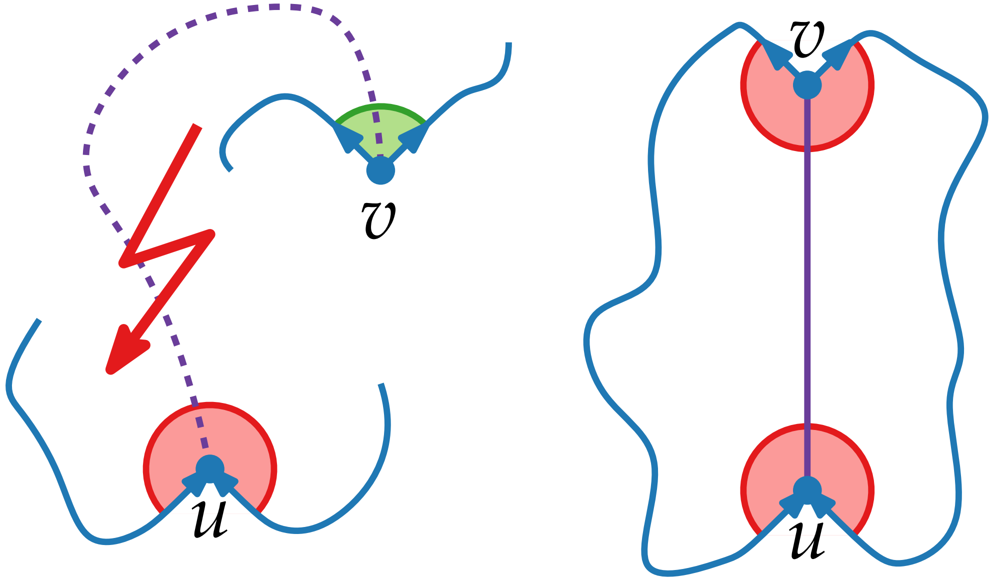

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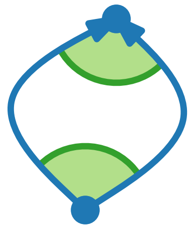
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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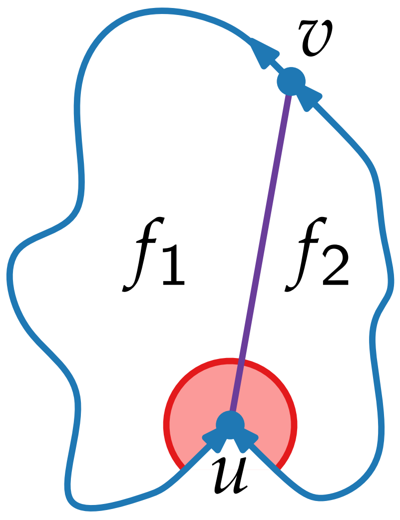

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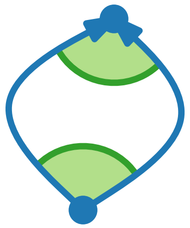
- $L(f) = 0$  $\Rightarrow S(f) = 2$
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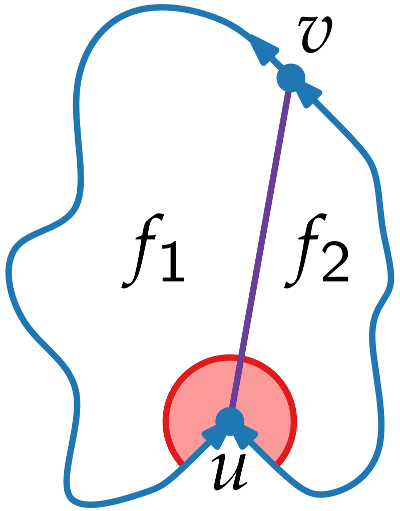

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- Otherwise "high" source u exists.

Number of large angles

Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source/sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$

Proof.
Observation and from Lemma 1: $L(f) + S(f) = 2A(f)$
and from Lemma 2: $L(f) - S(f) = \pm 2.$

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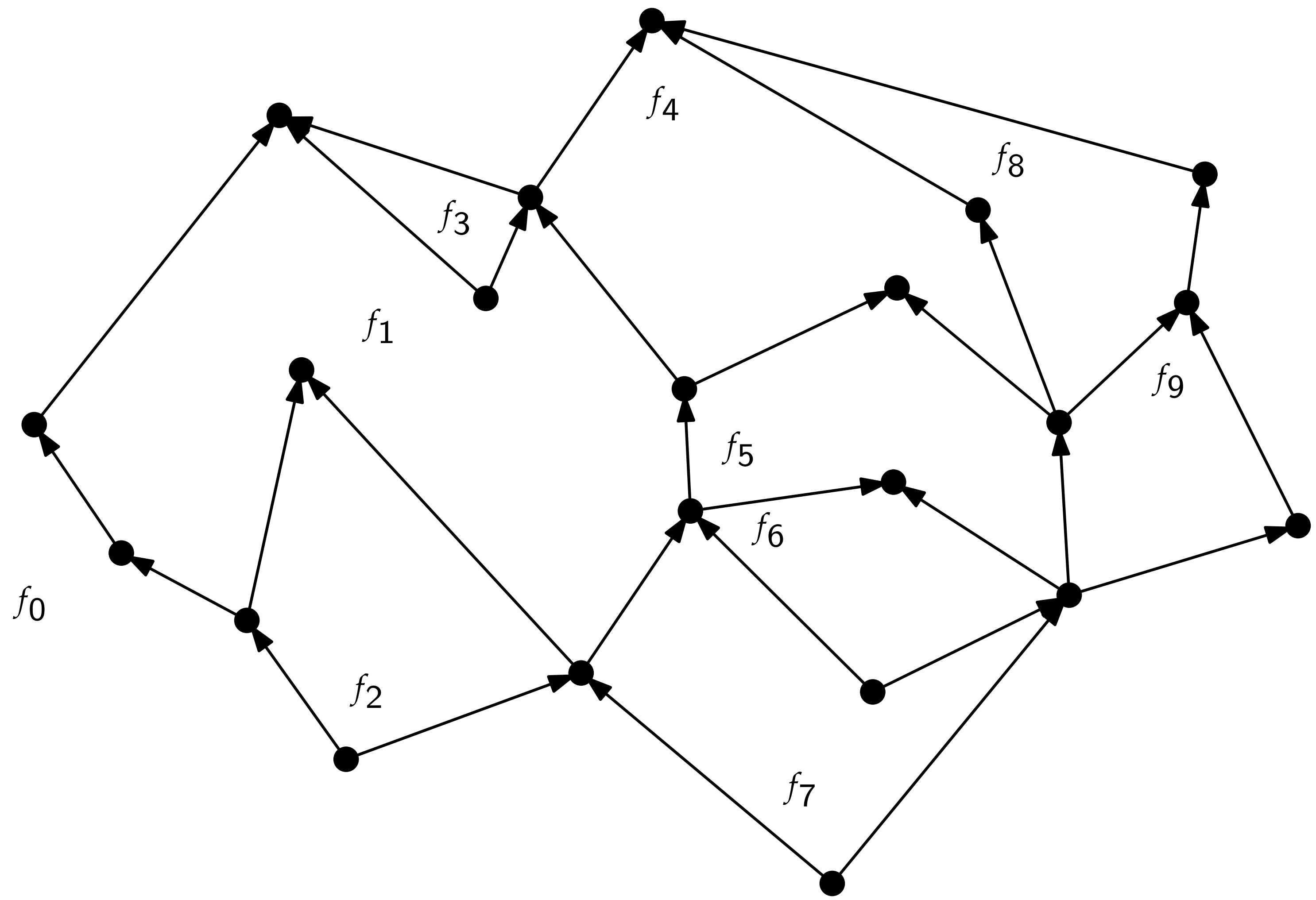

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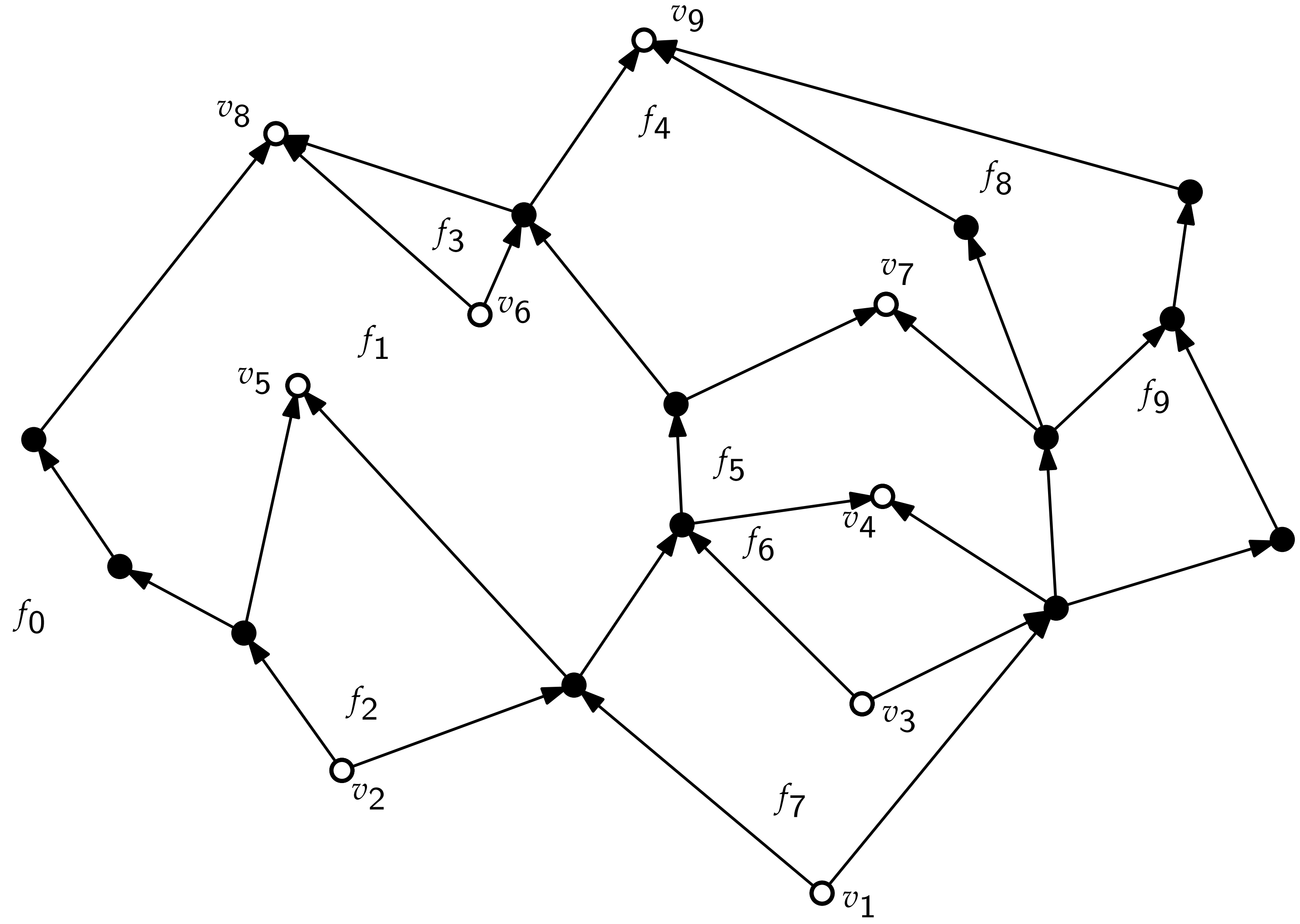

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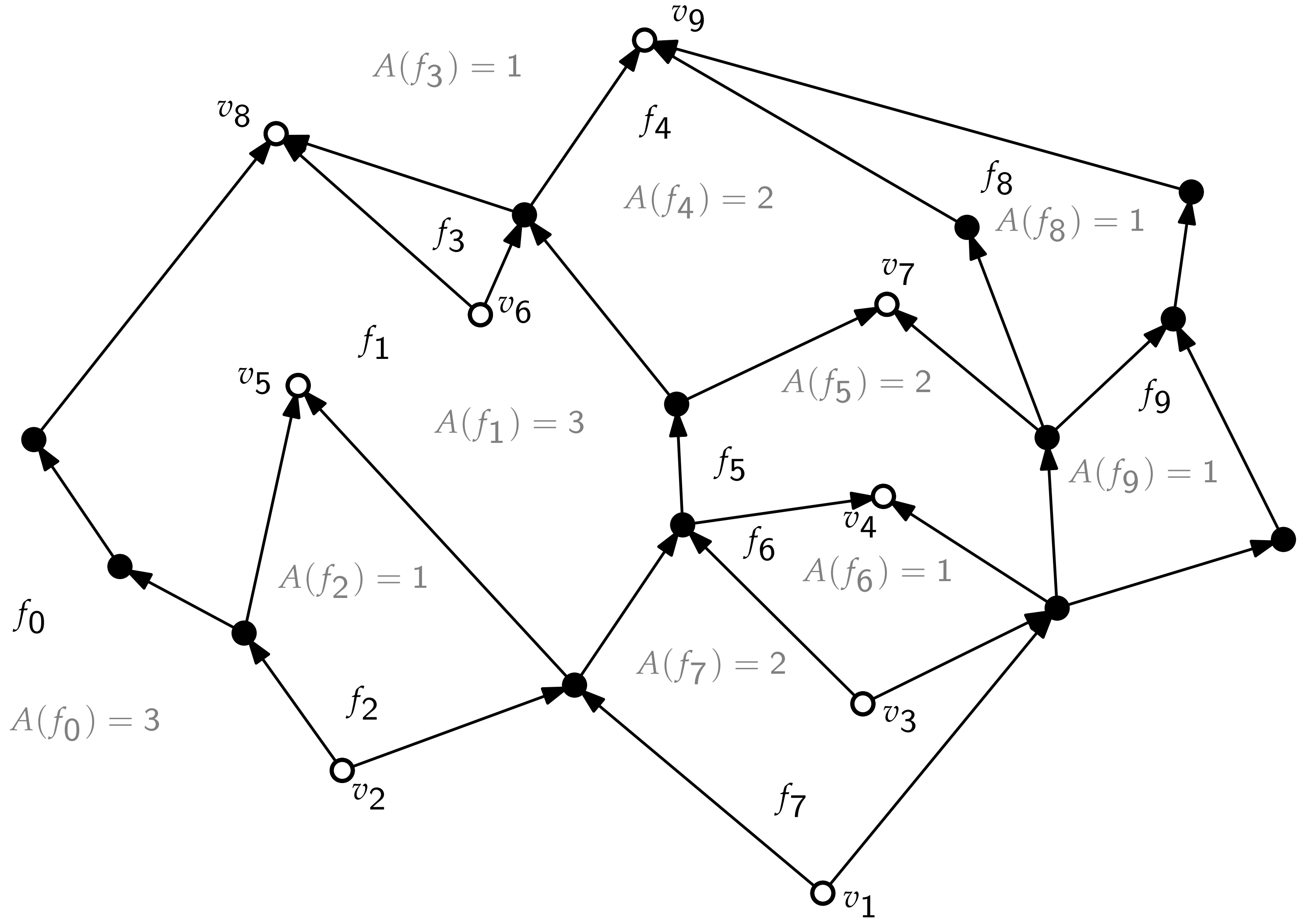

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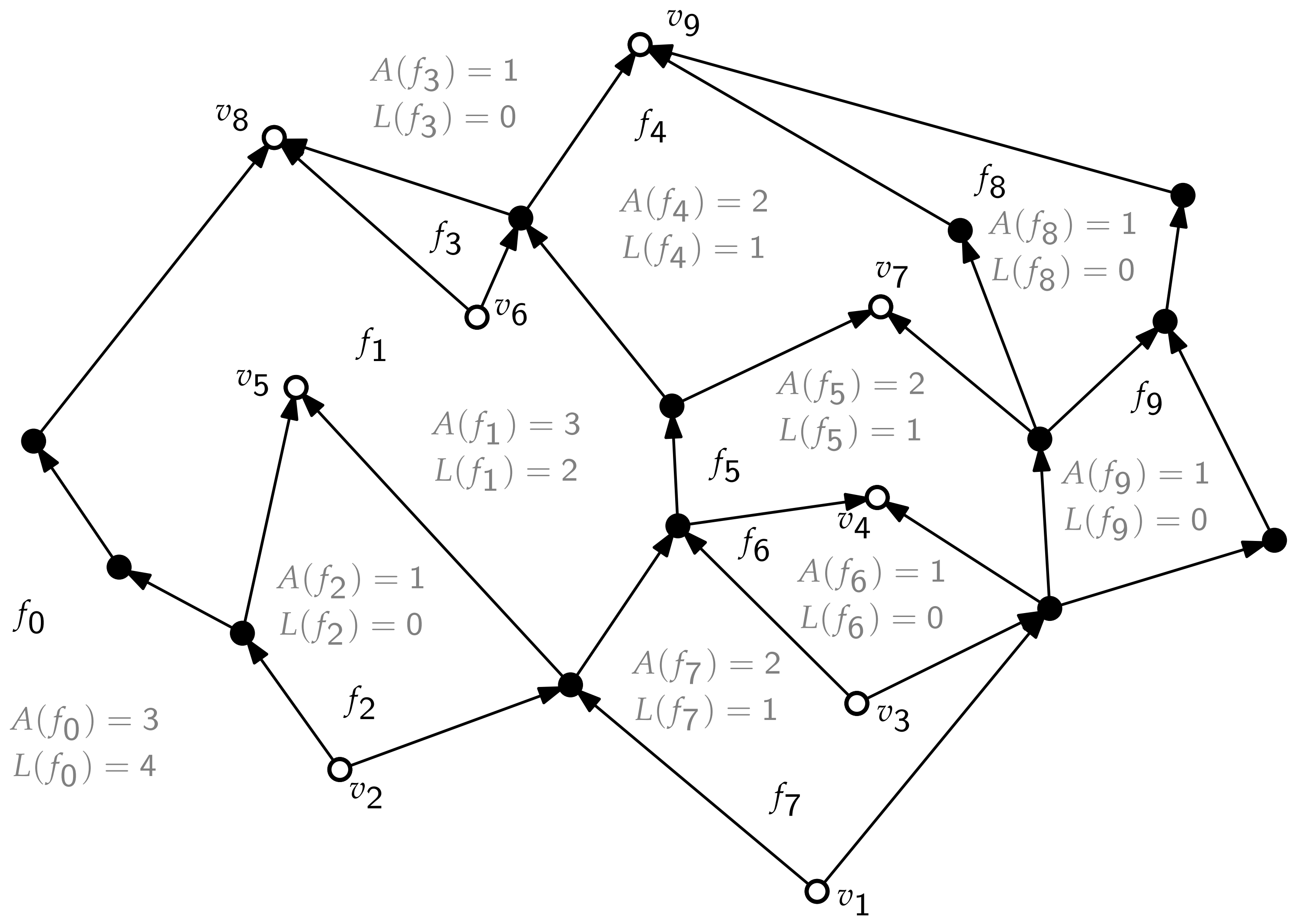

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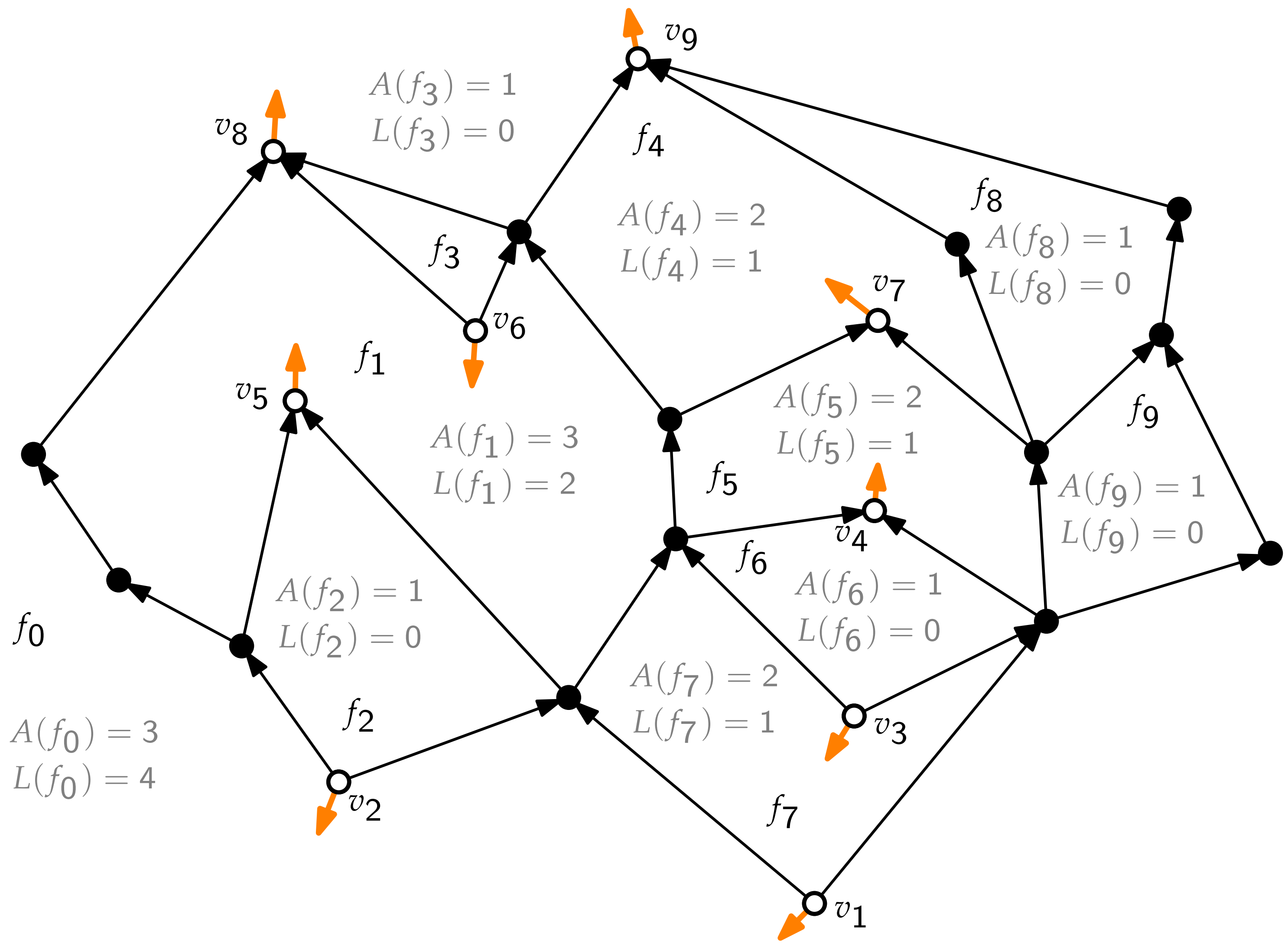

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- $A(f)$ # sources/sinks of f
- assignment $\Phi : S \cup T \to F$

Result characterisation

Theorem 3.
Let $G = (V, E)$ be an acyclic plane digraph with embedding given by F, f_0.
Then G is upward planar (respecting F, f_0) if and only if G is bimodal and there exists consistent assignment Φ.

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$\Leftarrow$: Idea:

- Construct planar st-digraph that is supergraph of G.
- Apply equivalence from Theorem 1.

Refinement algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

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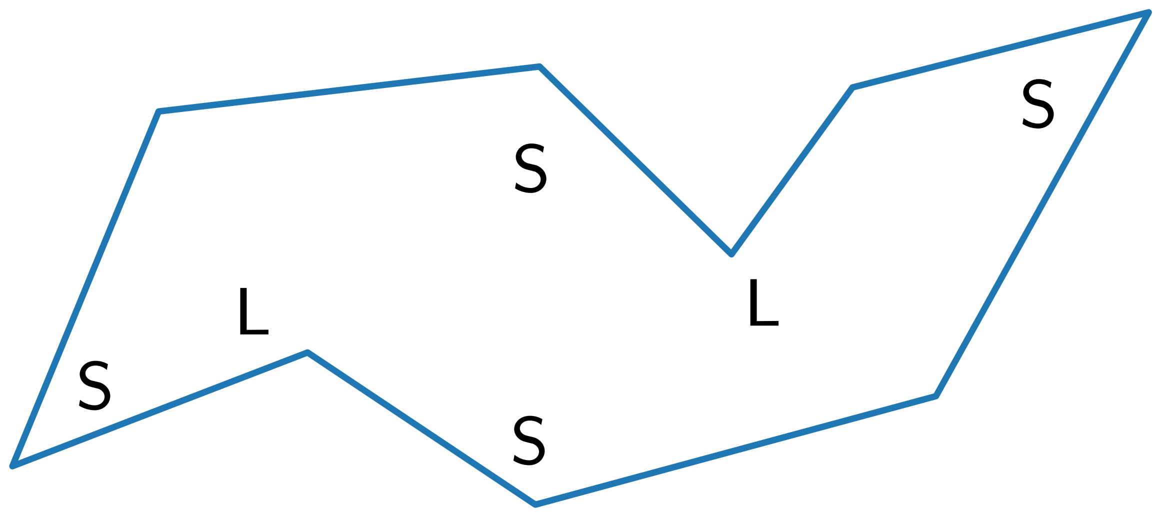

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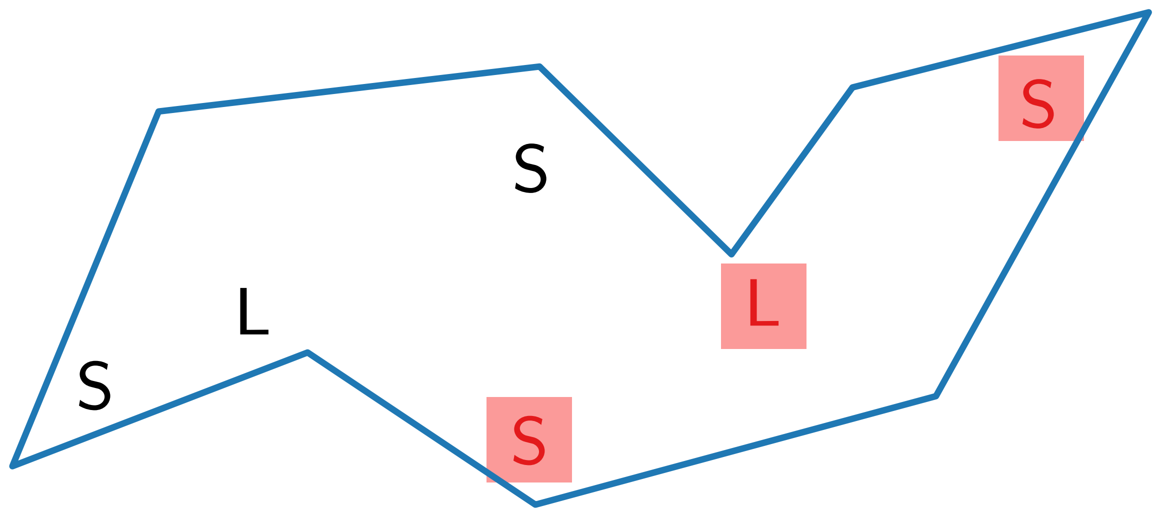

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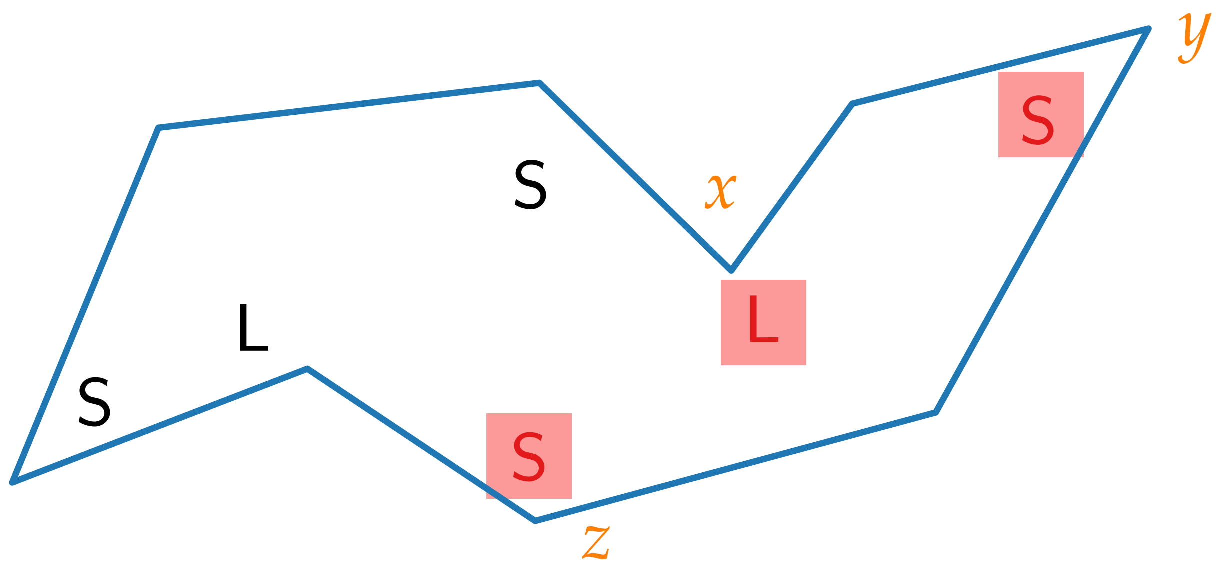

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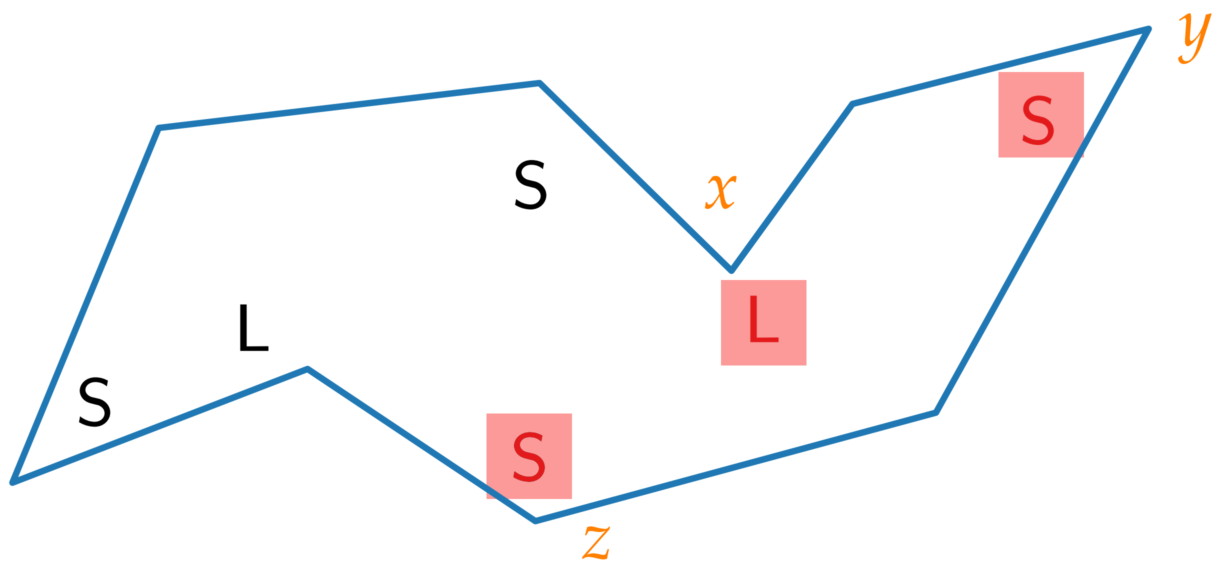

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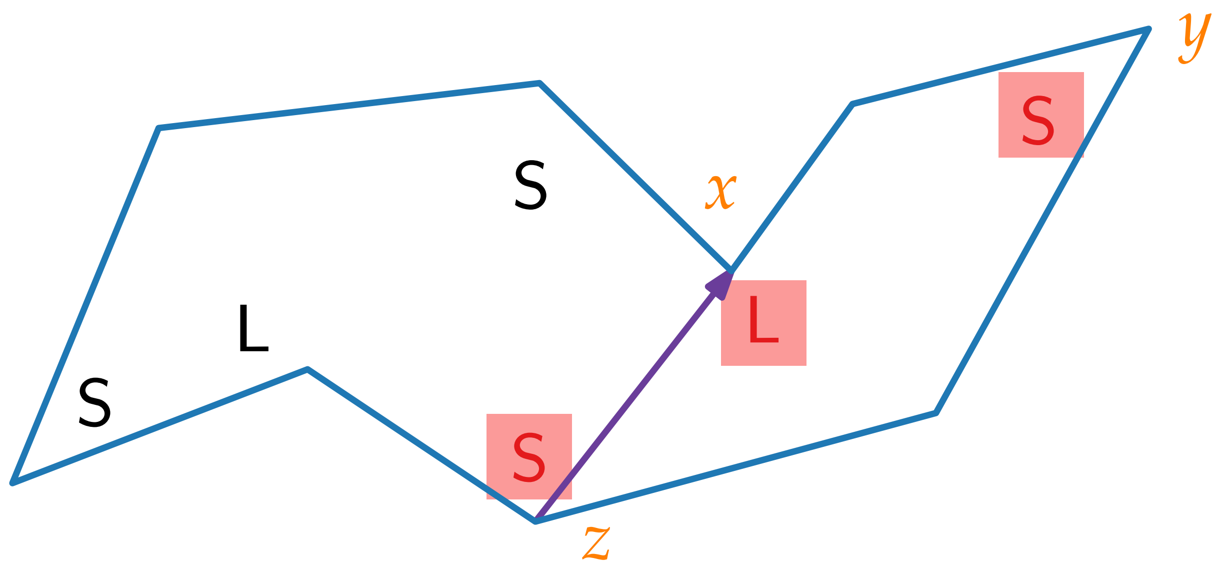

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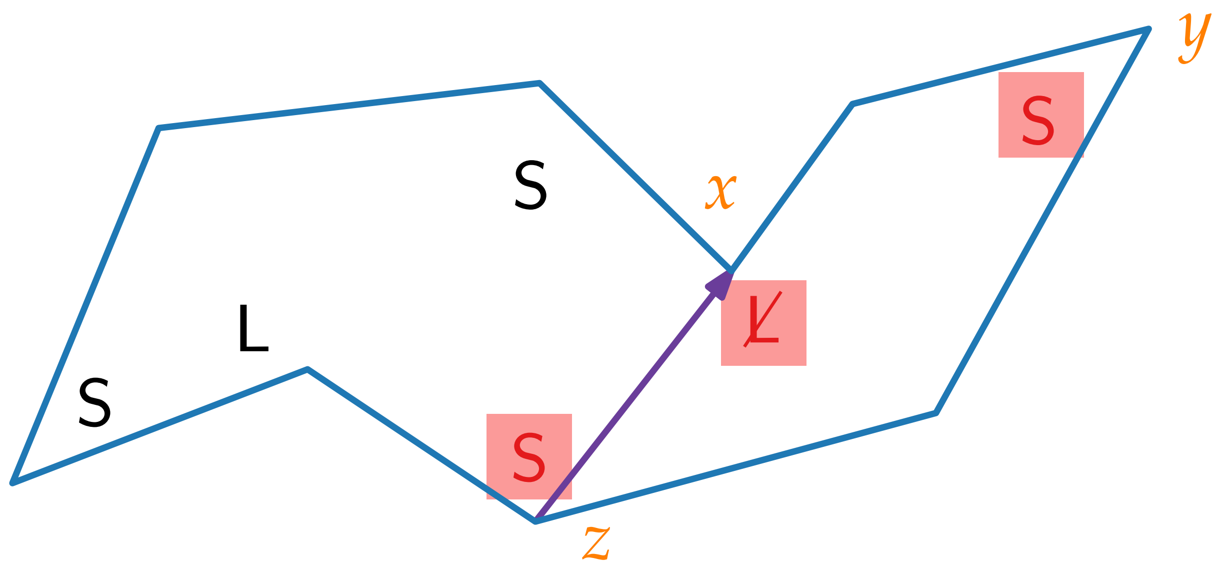

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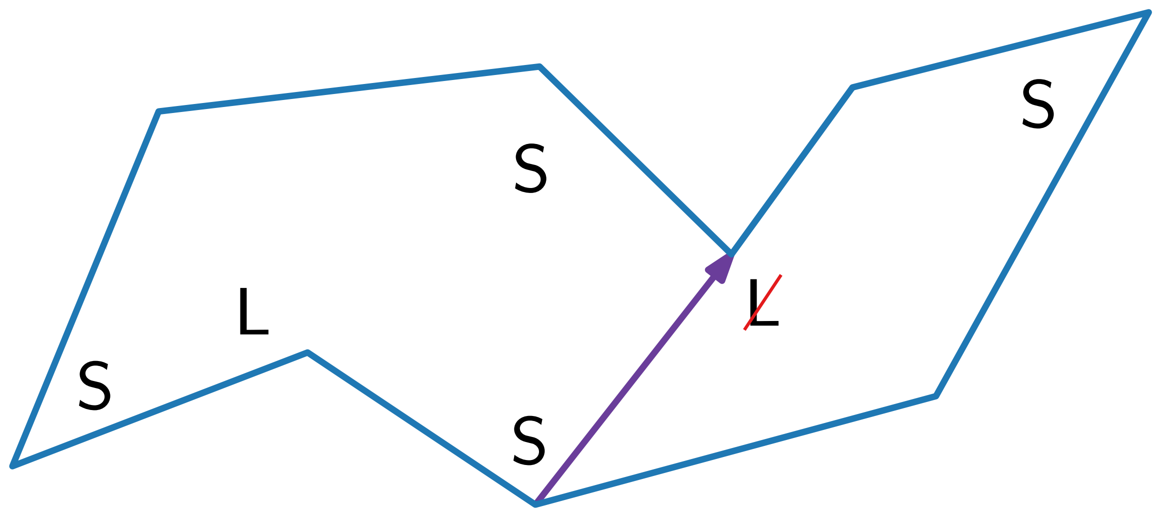

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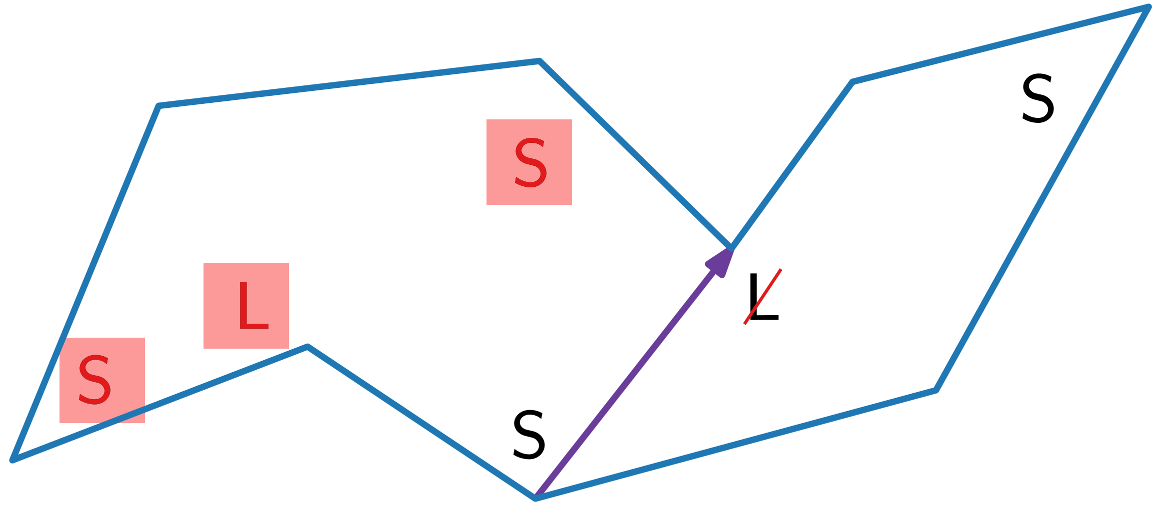

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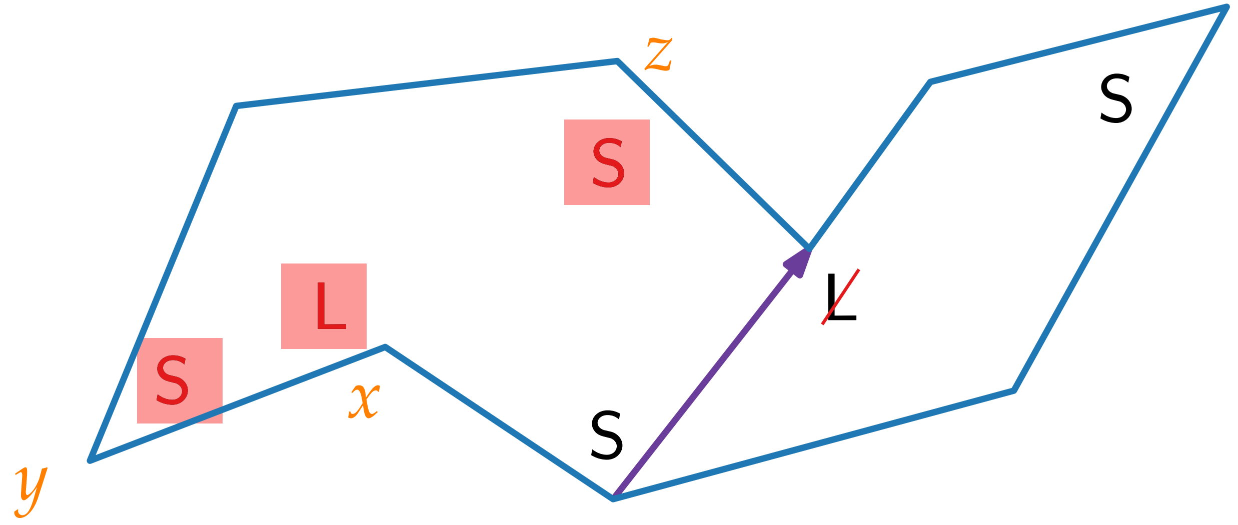

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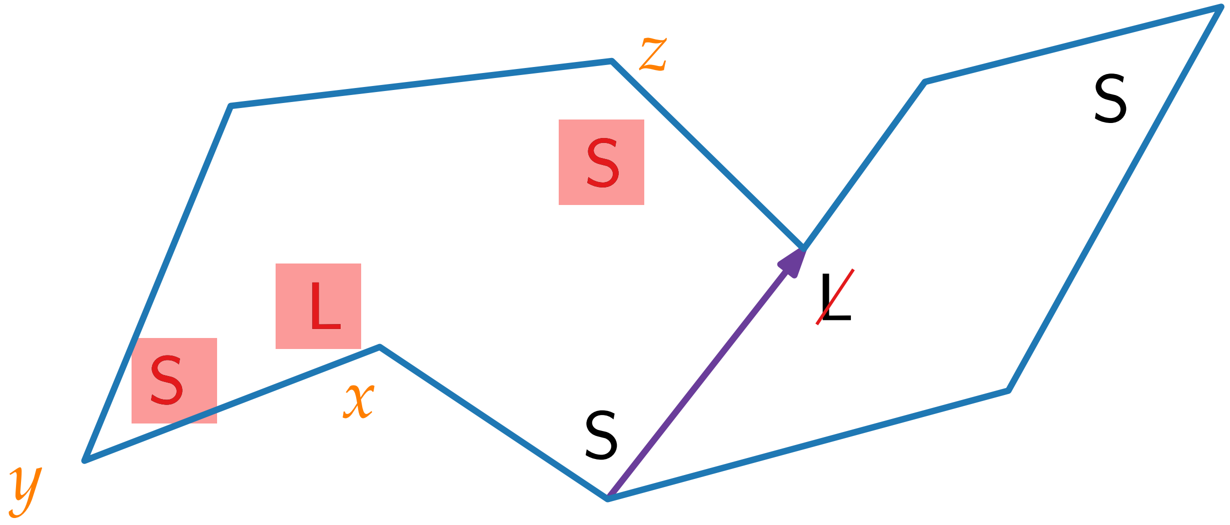

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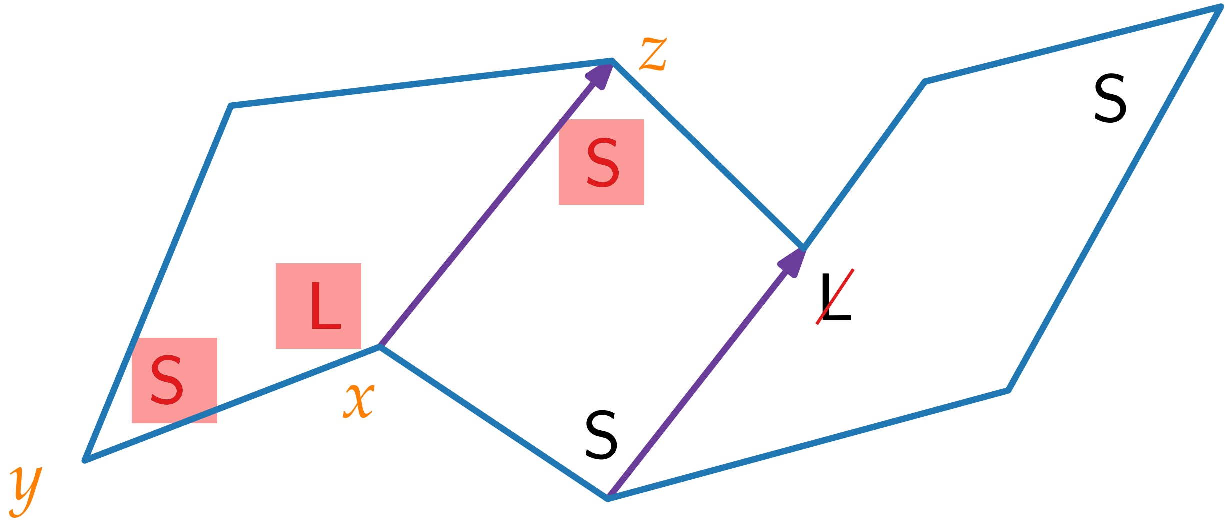

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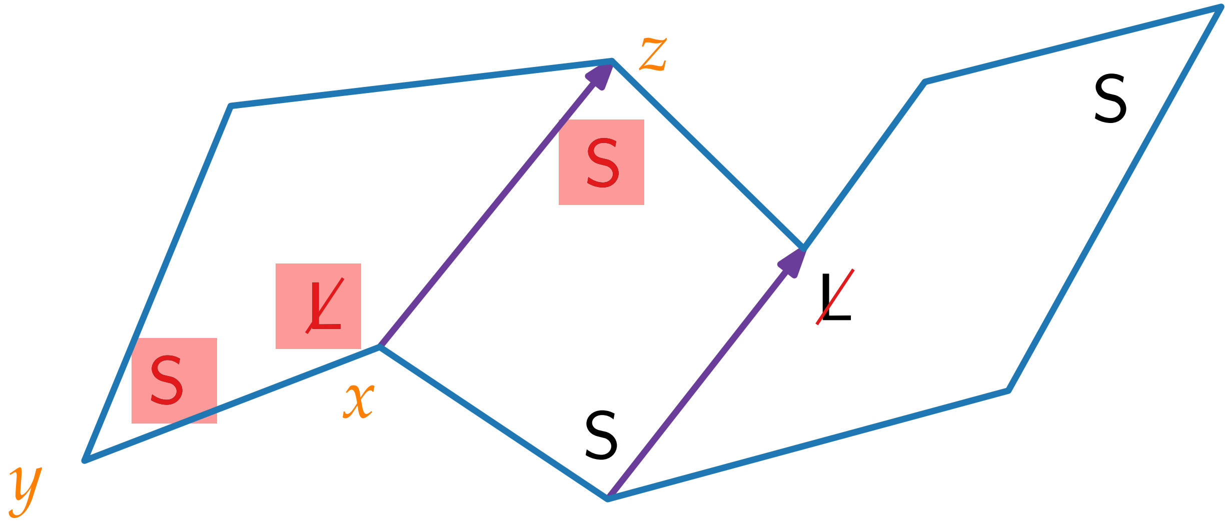

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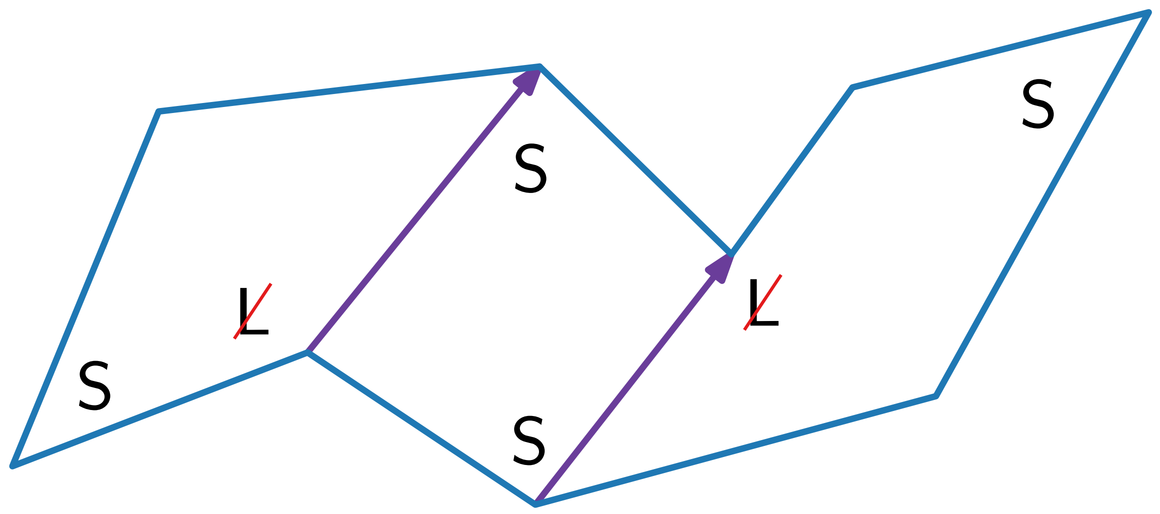

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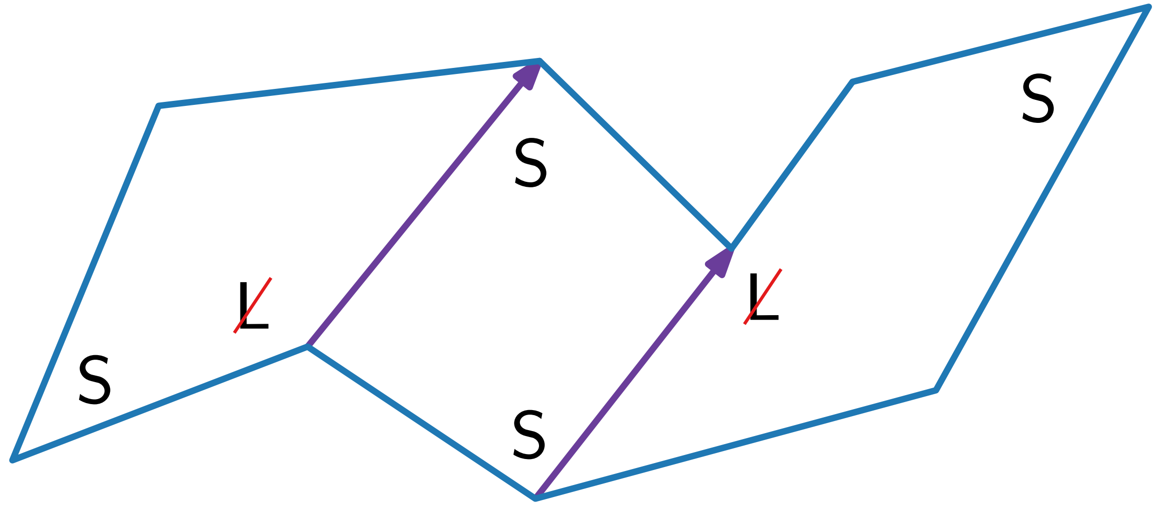

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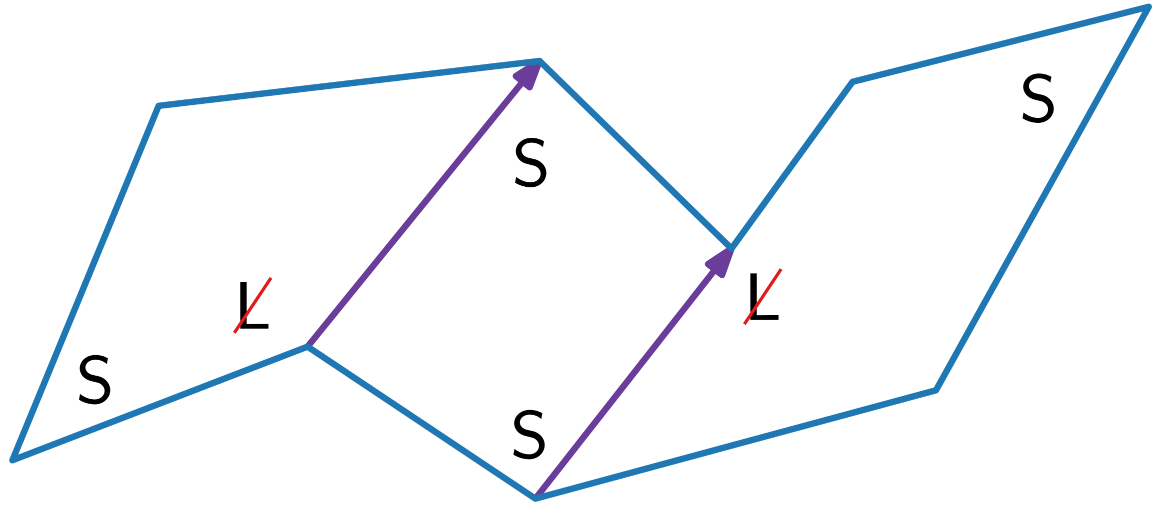
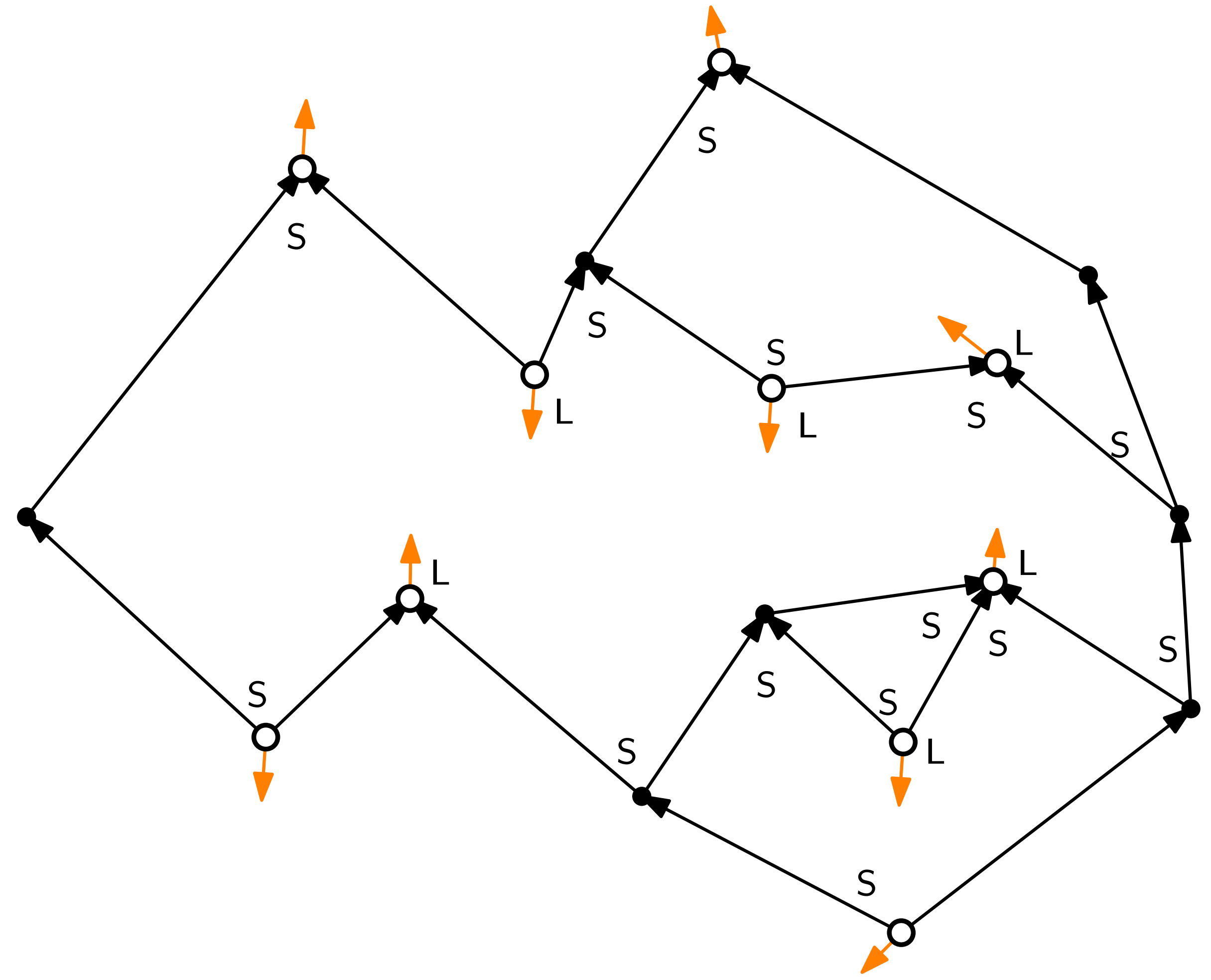


- Refine all faces. $\Rightarrow$ G is contained in a planar st-digraph.
- Planarity, acyclicity, bimodality are invariants under construction.

Refinement example



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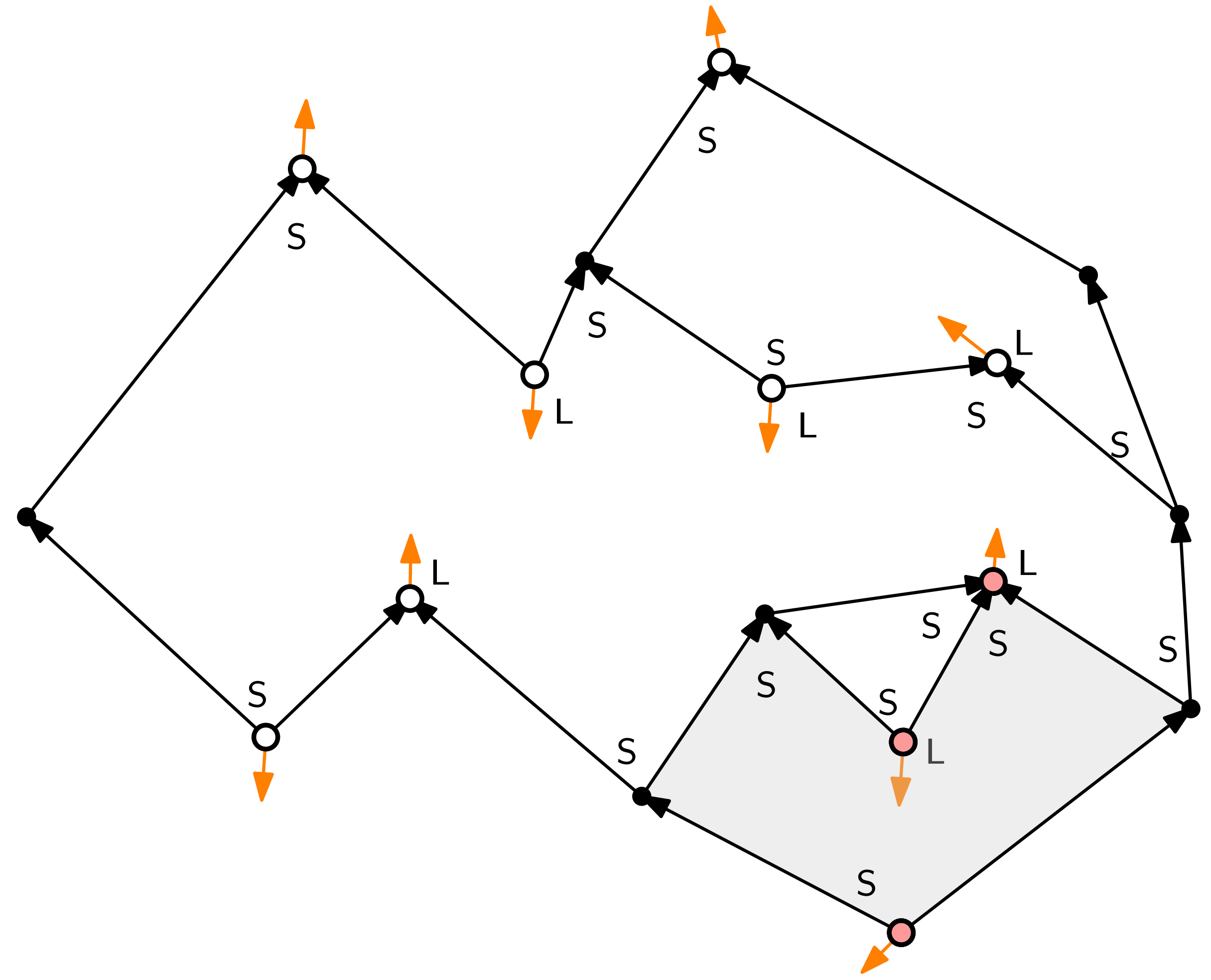

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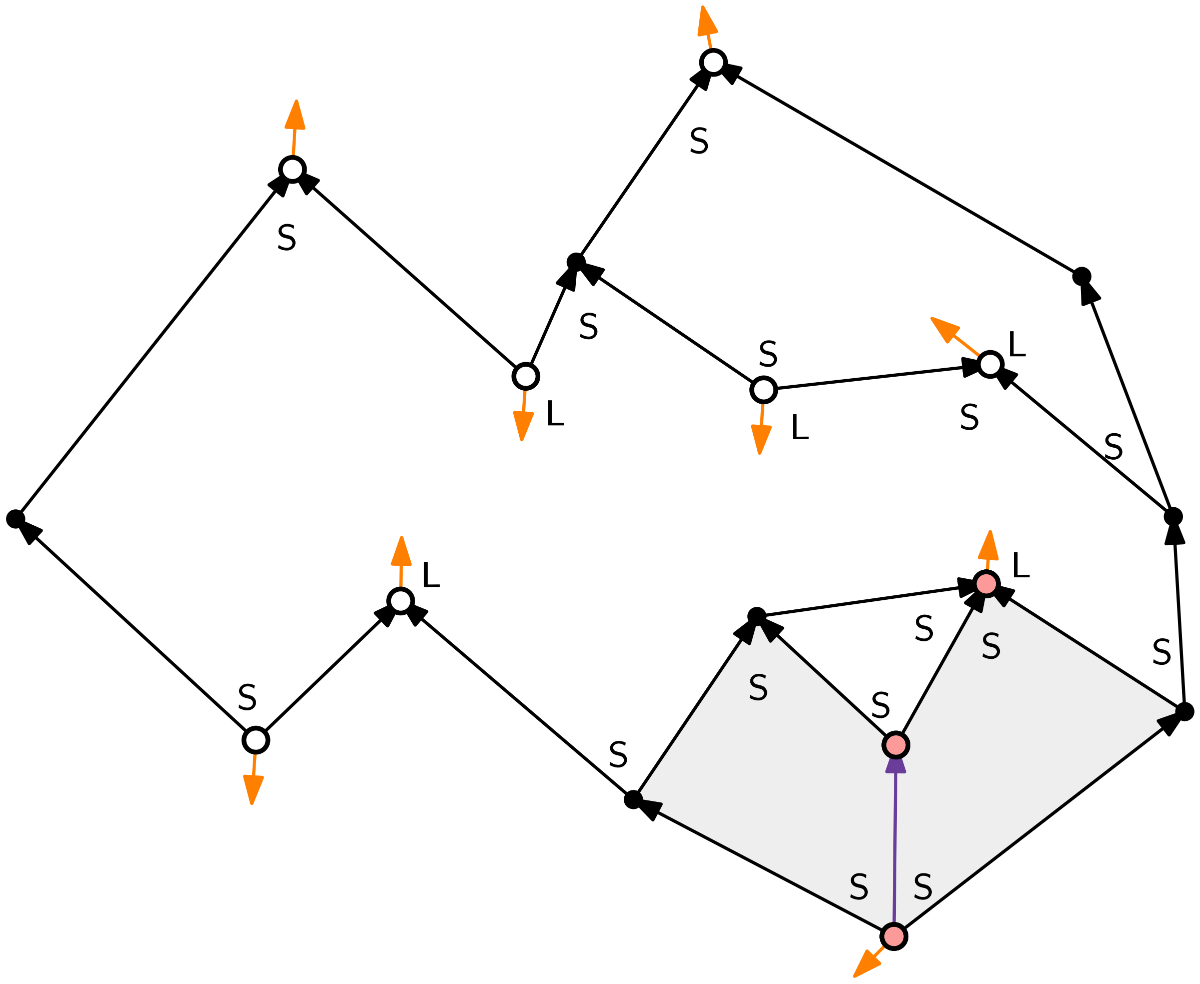

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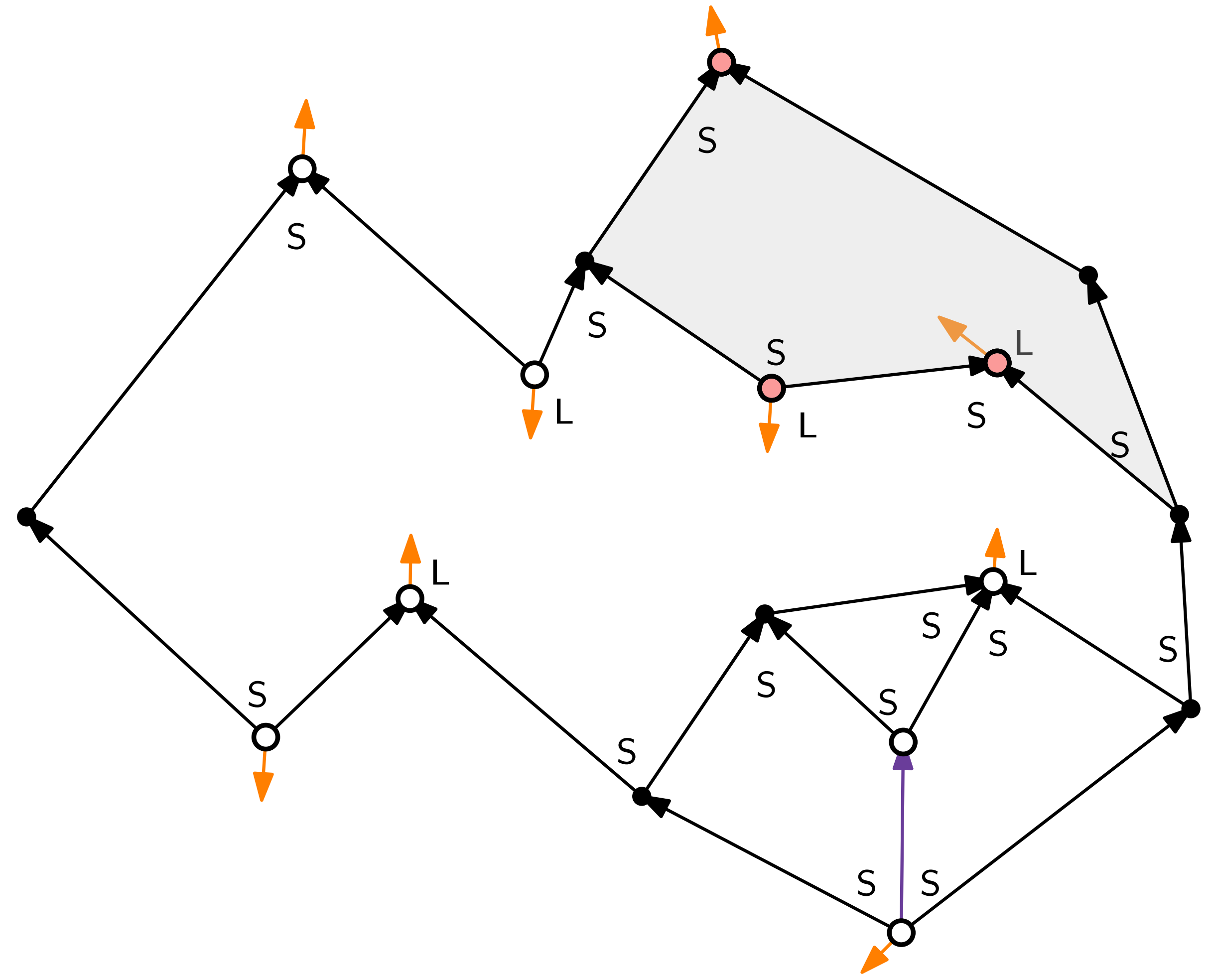

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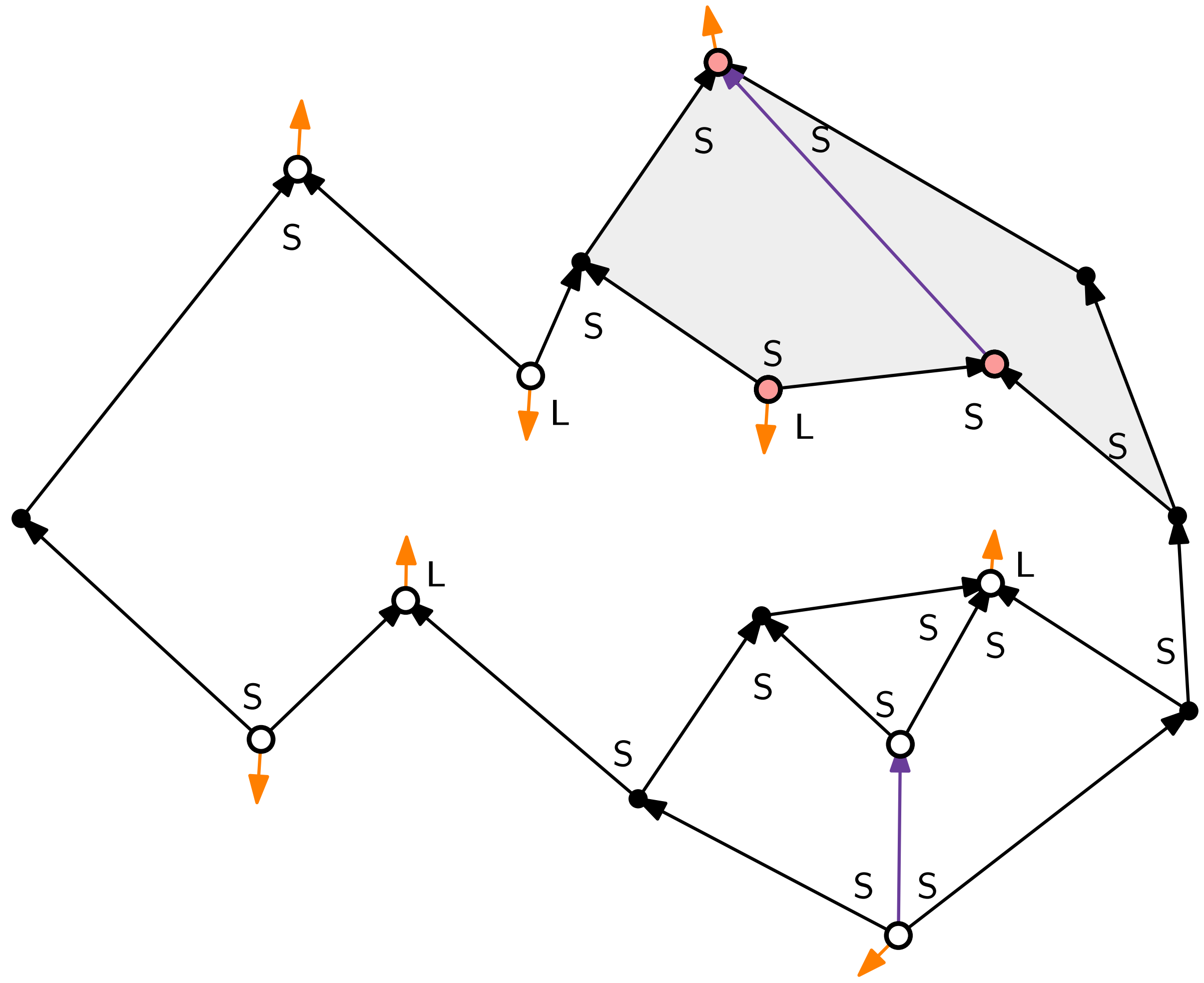

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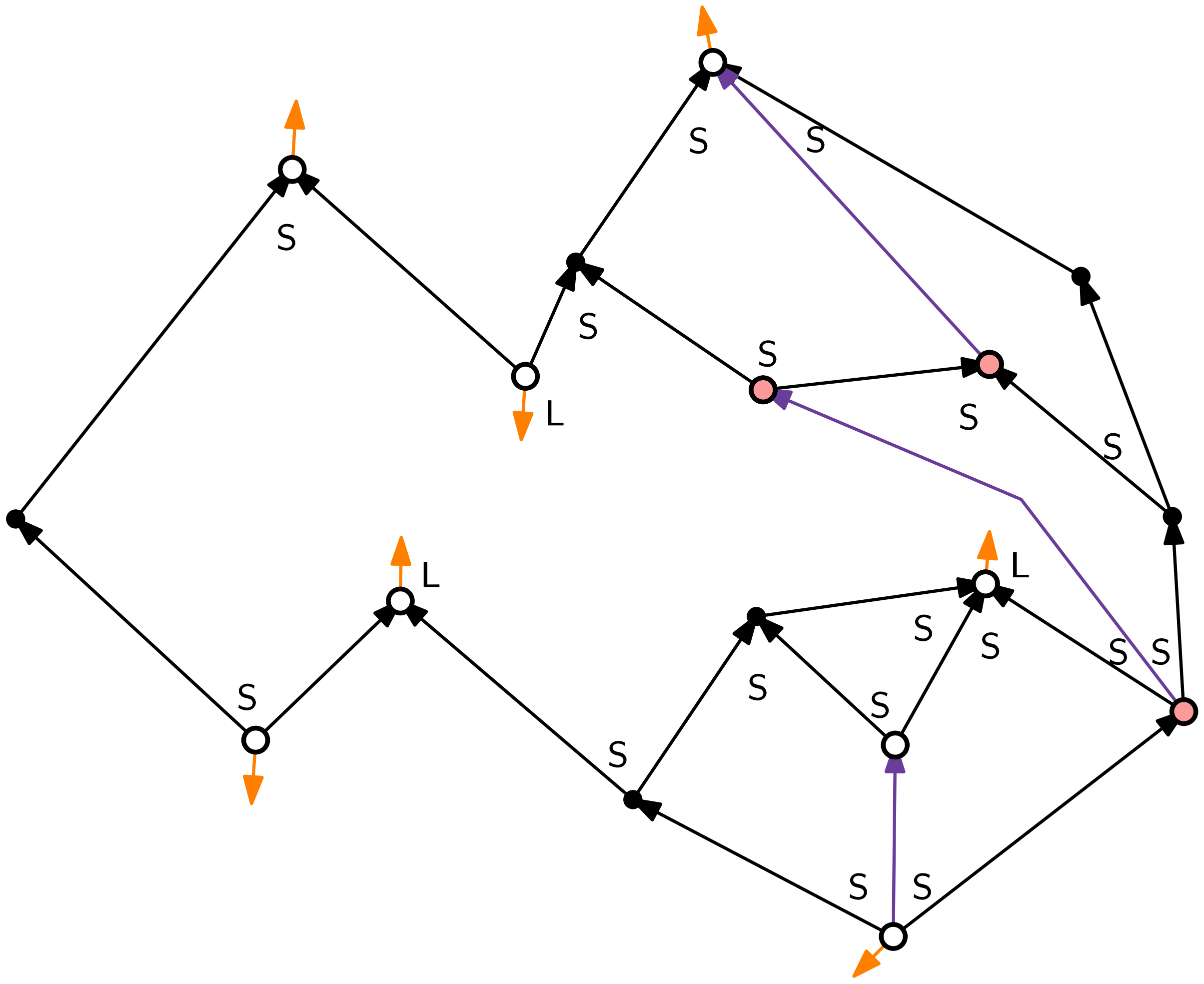

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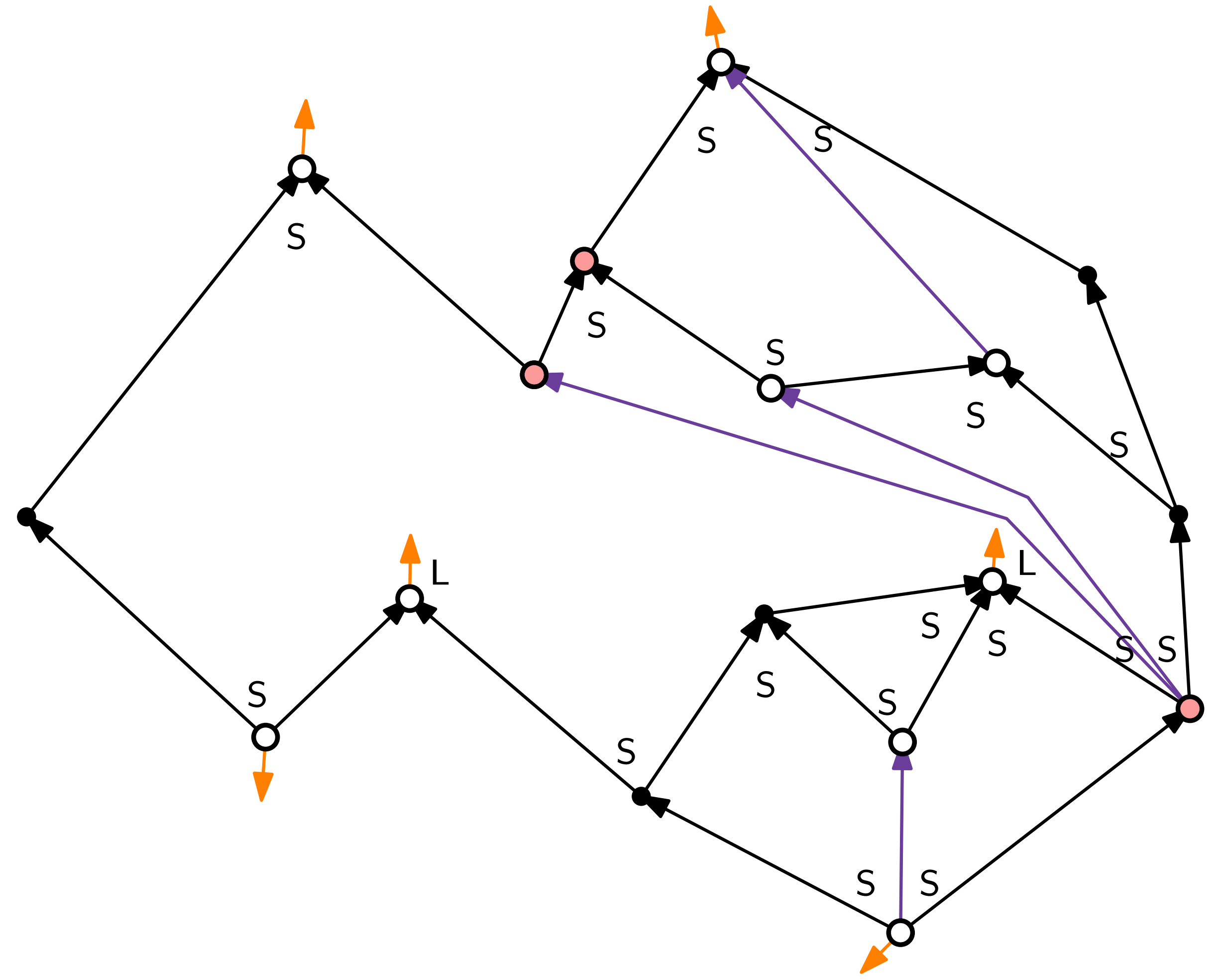

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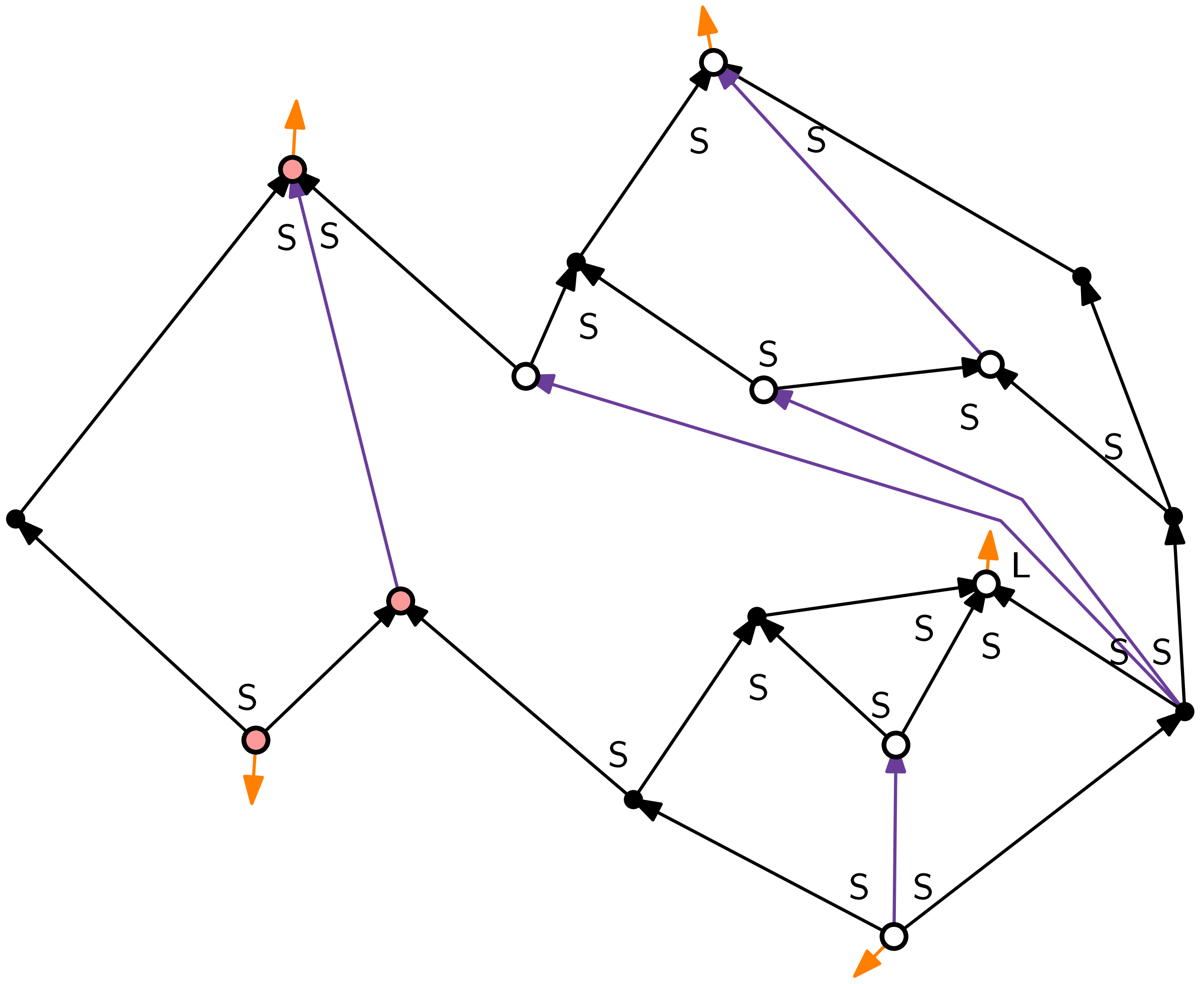

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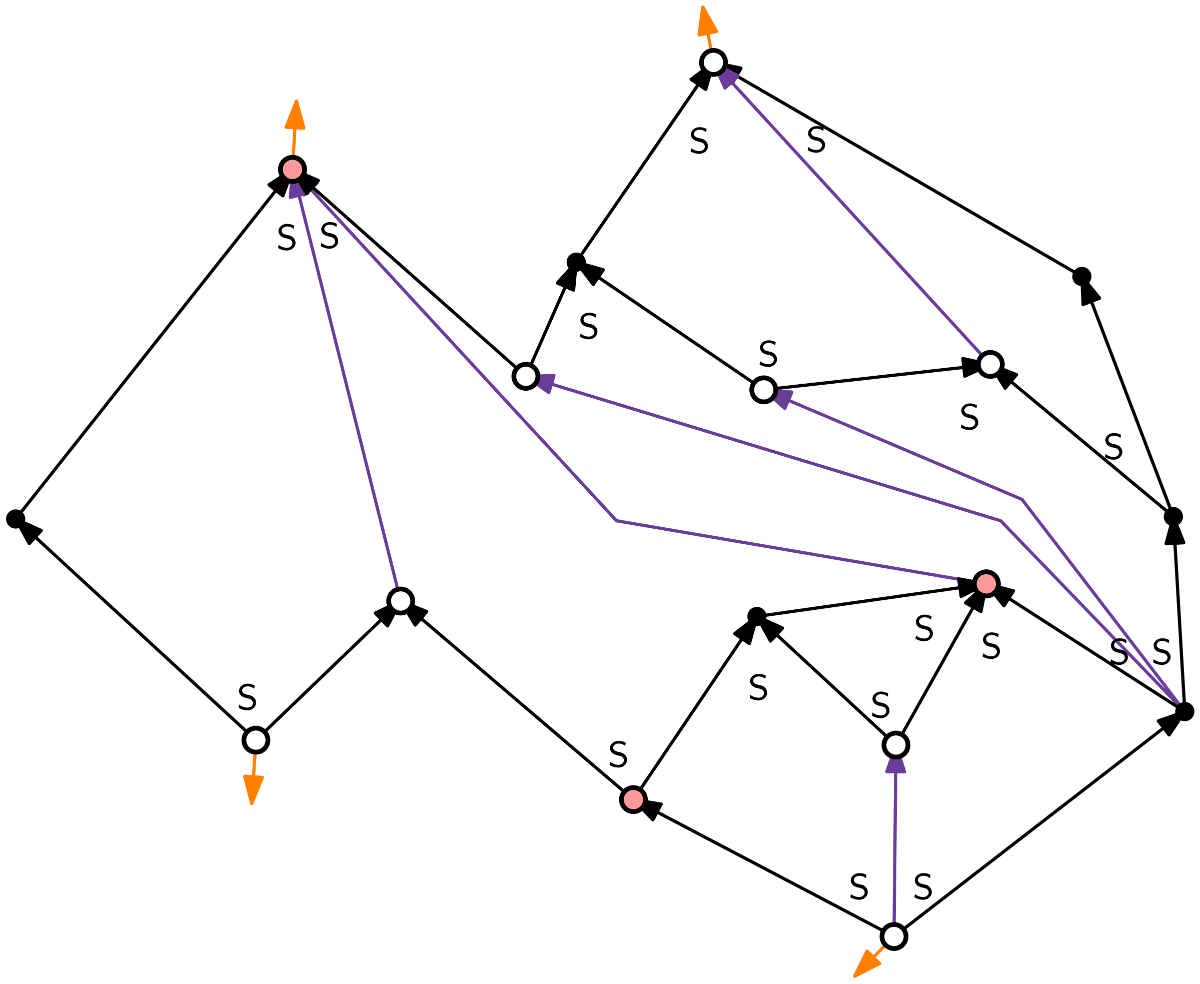

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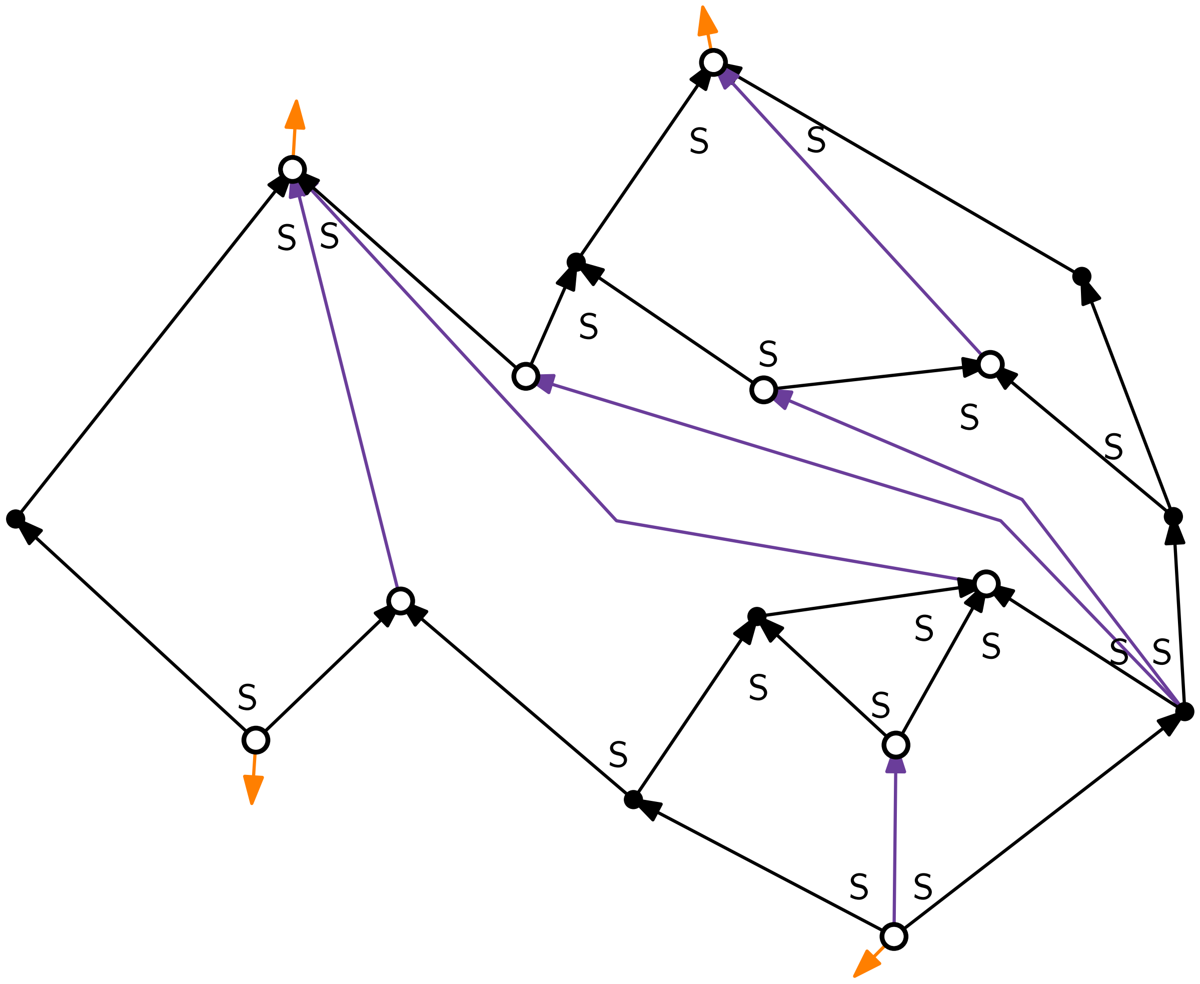

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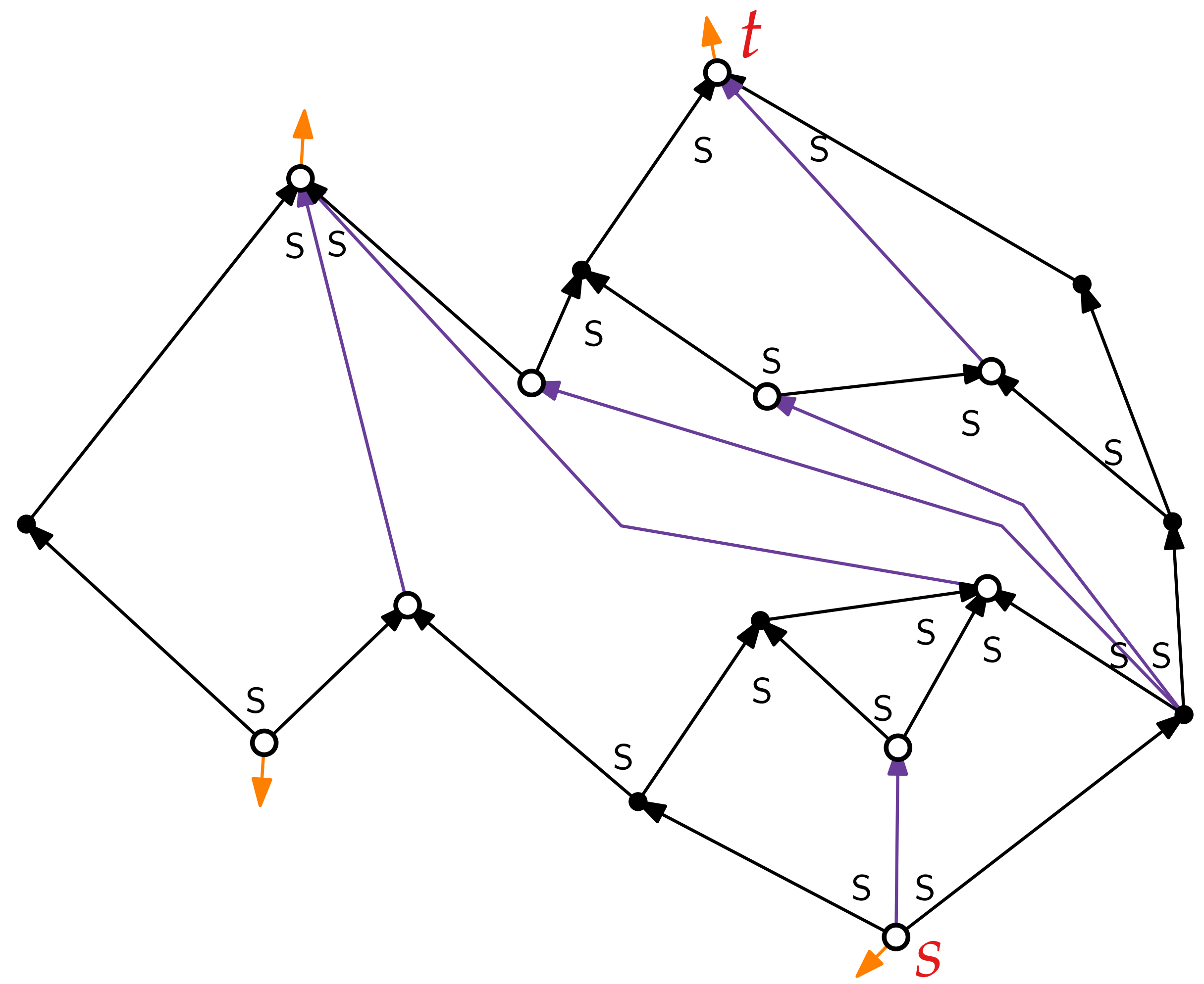

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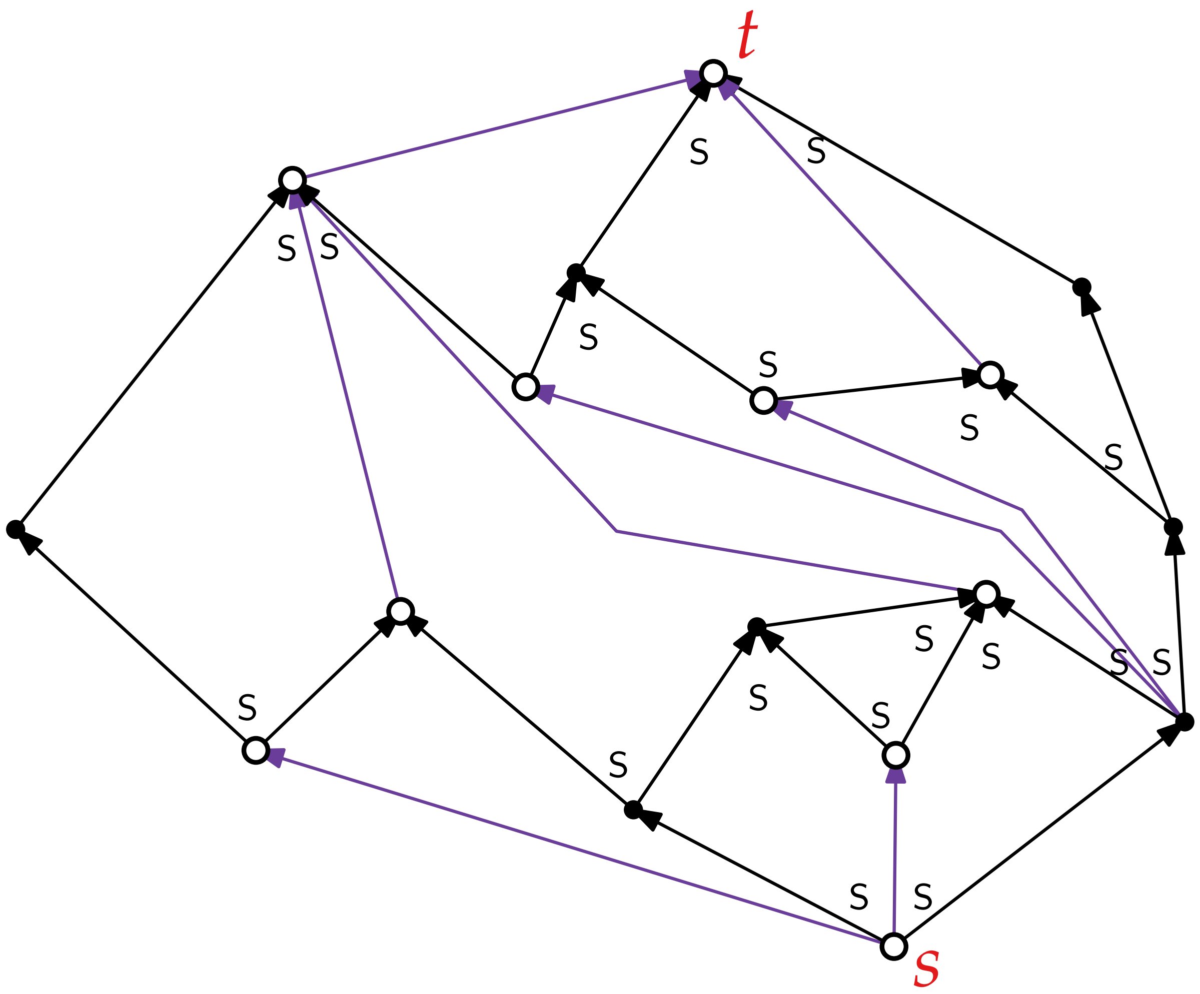

Result upward planarity test

Theorem 2. [Bertolazzi et al., 1994]
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- If G bimodal and Φ exists, refine G to plane st-digraph H.
- Draw H upward planar.
- Deleted edges added in refinement step.

Finding a consistent assignment

Idea.
Flow $(v, f) = 1$ from global source/sink v to the incident face f its large angle gets assigned to.

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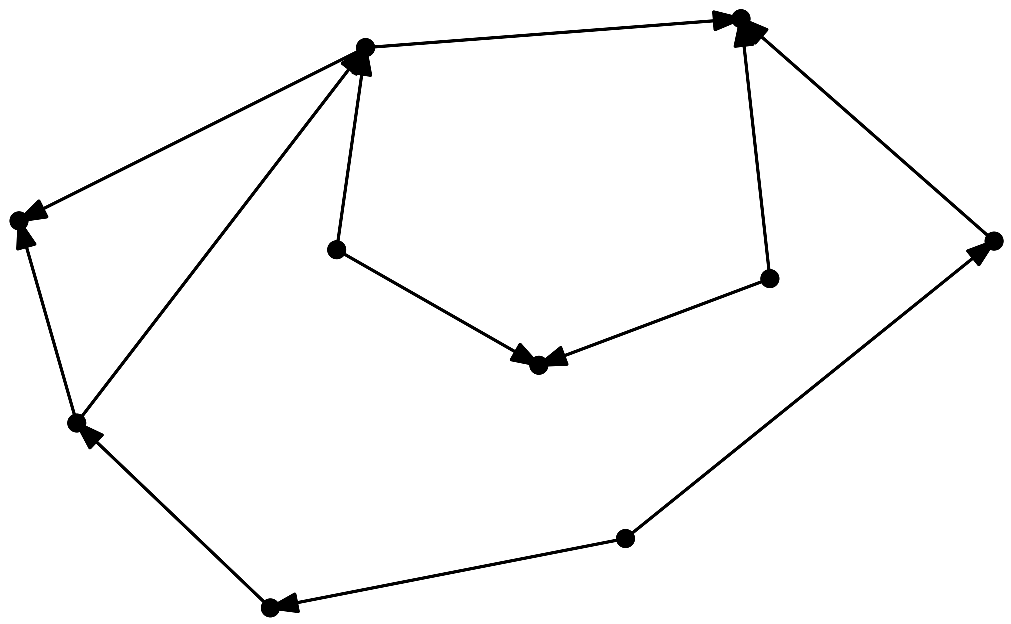

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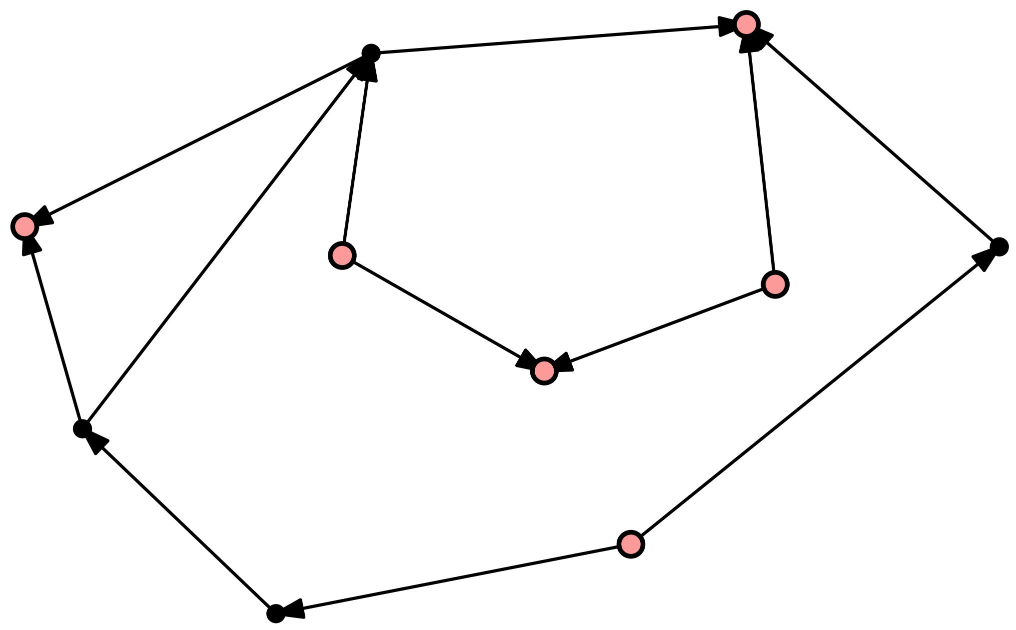

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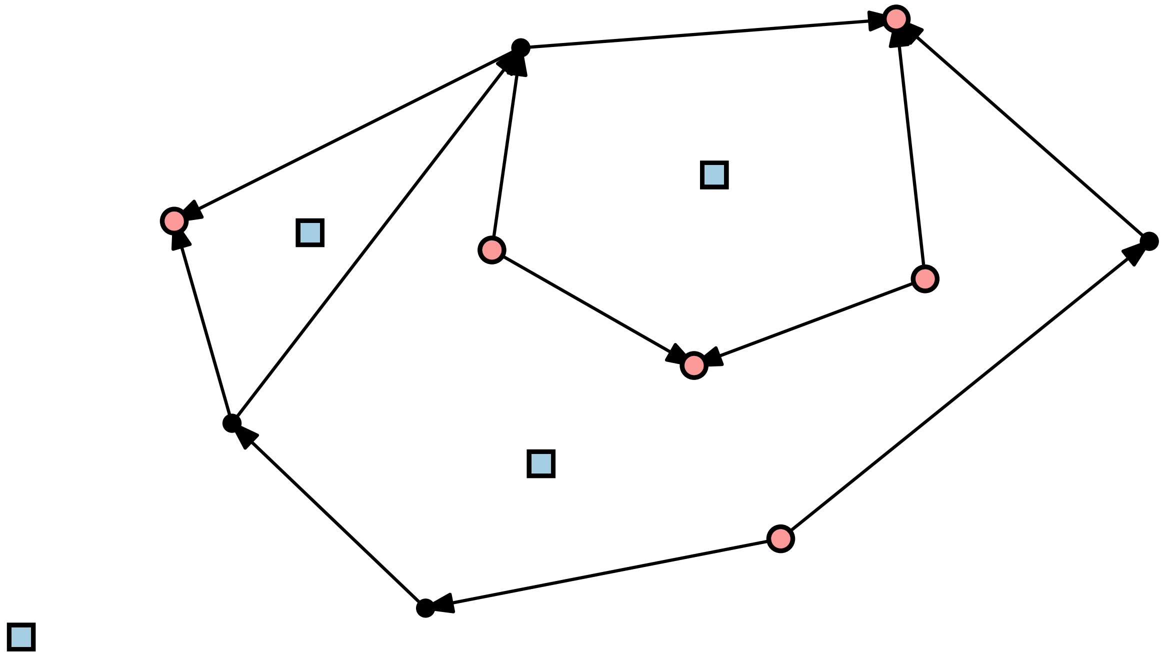

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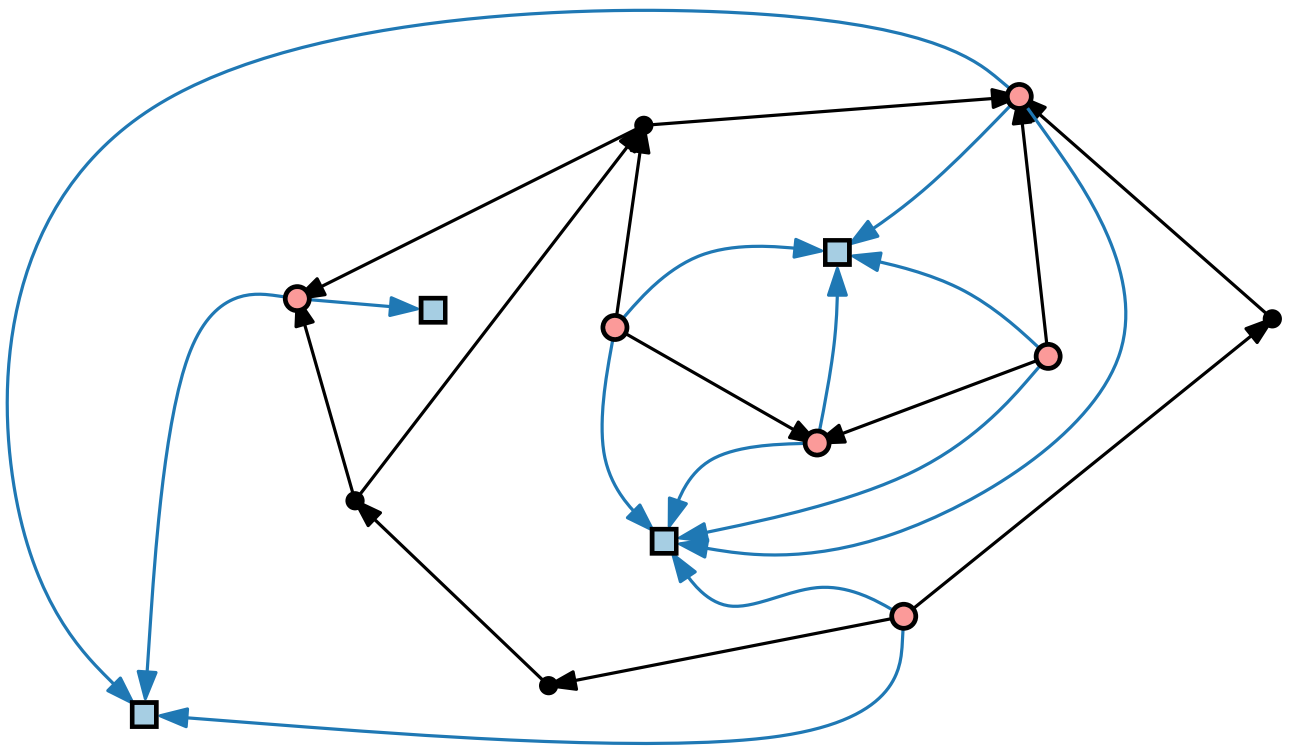

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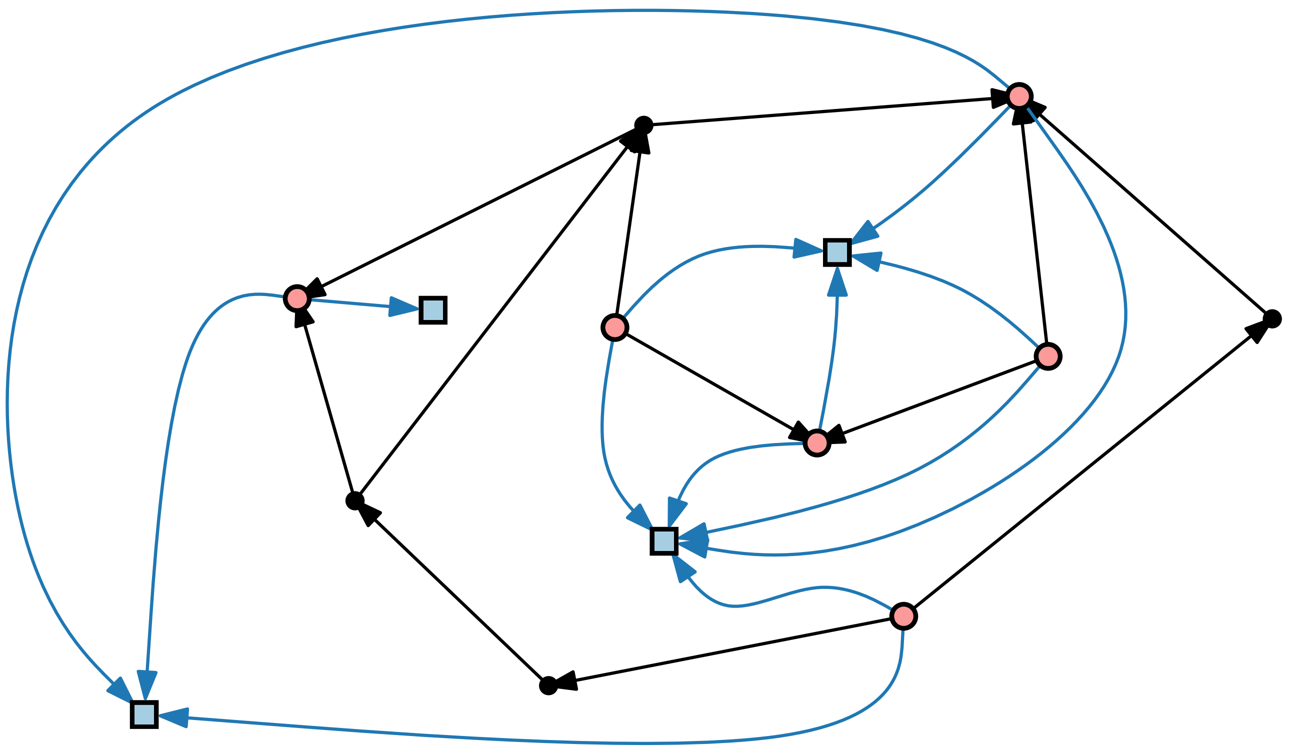

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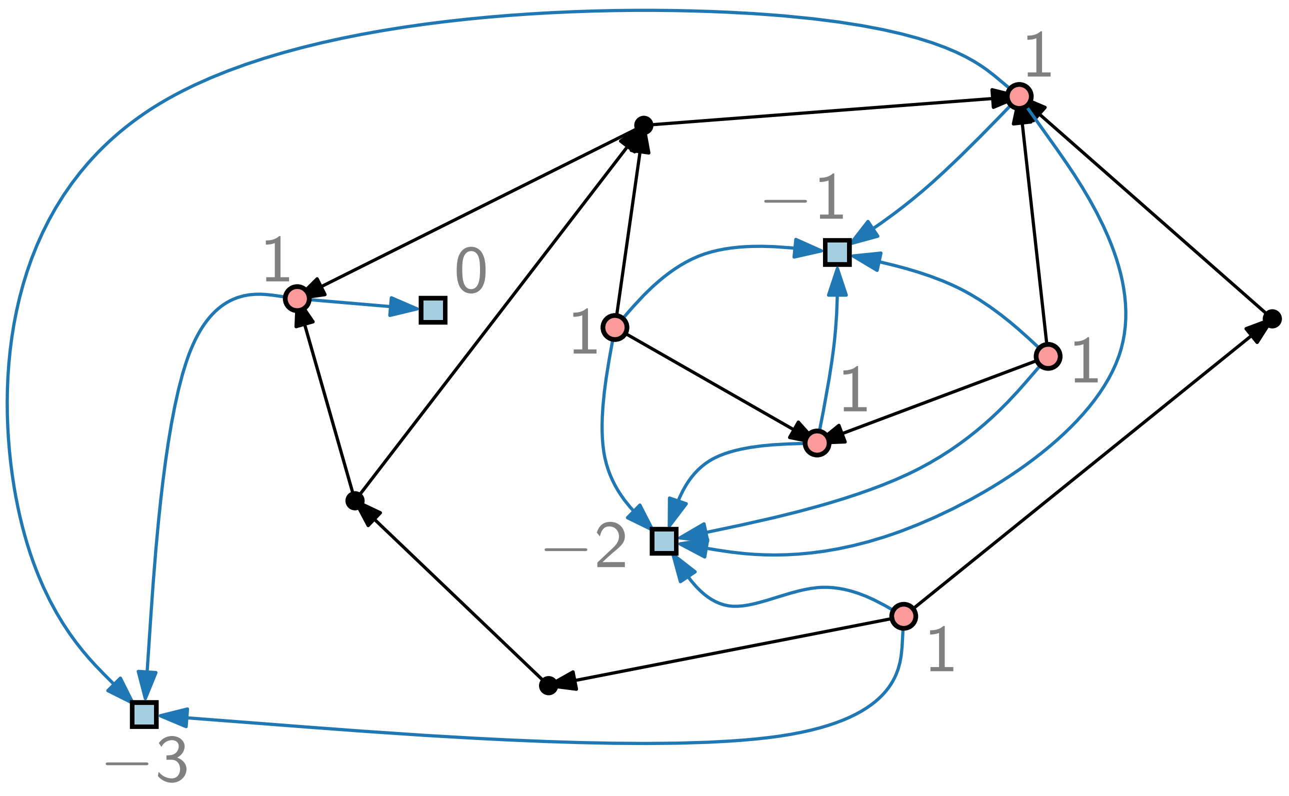

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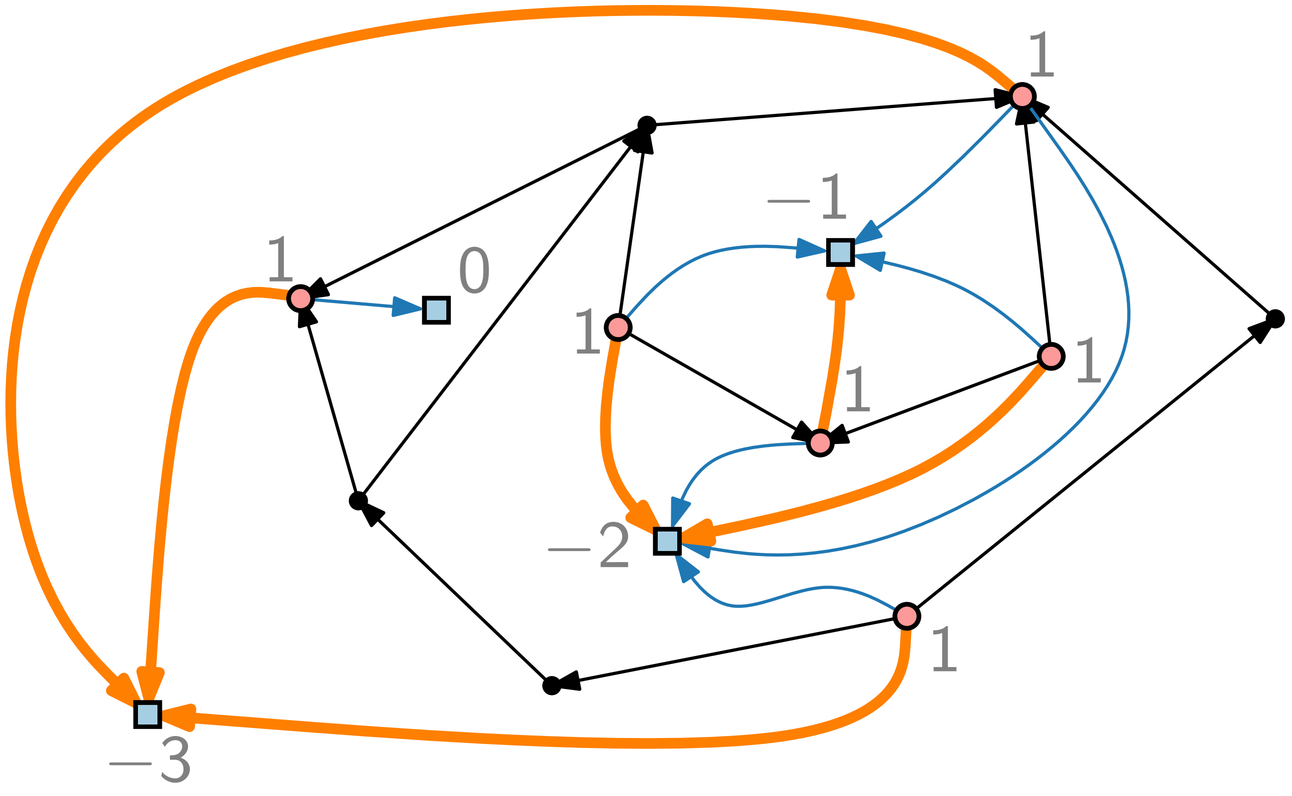

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- Many related concepts have been studied: quasi-planarity, upward drawings of mixed graphs, upward planarity on cyclinder/torus, ...

Literature

- [GD Ch. 6] for detailed explanation

Orginal papers referenced:

- [Kelly '87] Fundamentals of Planar Ordered Sets
- [Di Battista, Tamassia '88] Algorithms for Plane Representations of Acyclic Digraphs
- [Garg, Tamassia '95] On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Hutton, Lubiw '96] Upward Planar Drawing of Single-Source Acyclic Digraphs
- [Bertolazzi, Di Battista, Mannino, Tamassia '94] Upward Drawings of Triconnected Digraphs
- [Healy, Lynch '05] Building Blocks of Upward Planar Digraphs
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